Discrete parabolic groups in PSL(3, ℂ)

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A R T I C L E   I N F O

A B S T R A C T

We study and classify the purely parabolic discrete subgroups of PSL(3, ℂ). This includes all discrete subgroups of the Heisenberg group Heis(3, ℂ). While for PSL(2, ℂ) every purely parabolic subgroup is Abelian and acts on $\mathbb{P}^1_\mathbb{C}$ with limit set a single point, the case of PSL(3, ℂ) is far more subtle and intriguing. We show that there are five families of purely parabolic discrete groups in PSL(3, ℂ), and some of these actually split into subfamilies. We classify all these by means of their limit set and the control group. We use first the Lie-Kolchin Theorem and Borel's fixed point theorem to show that all purely parabolic discrete groups in PSL(3, ℂ) are virtually triangularizable. Then we prove that purely parabolic groups in PSL(3, ℂ) are virtually solvable and polycyclic, hence finitely presented. We then prove a slight generalization of the Lie-Kolchin Theorem for these groups: they are either virtually unipotent or else Abelian of rank 2 and of a very special type. All the virtually unipotent ones turn out to be conjugate to subgroups of the Heisenberg group Heis(3, ℂ). We classify these using the obstructor dimension introduced by Bestvina, Kapovich and Kleiner. We find that their Kulkarni limit set is either a projective line, a cone of lines with base a circle or else the whole $\mathbb{P}^2_\mathbb{C}$. We determine the relation with the Conze-Guivarc'h limit set of the action on the dual projective space $\check{\mathbb{P}}^2_\mathbb{C}$ and we show that in all cases

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the Kulkarni region of discontinuity is the largest open set
where the group acts properly discontinuously.

Introduction

Henri Poincaré introduced in [26] the concept of Kleinian groups, *i.e. discrete subgroups of the Möbius group* Möb$(2, \mathbb{C})$, which is isomorphic to PSL$(2, \mathbb{C})$, and he classified its elements into three types. Poincaré's classification can be stated by saying that $g \in$ PSL$(2, \mathbb{C})$ is elliptic if it has a lift $\tilde{g}$ to SL$(2, \mathbb{C})$ which is diagonalizable and its eigenvalues are all unitary; g is parabolic if $\tilde{g}$ is non-diagonalizable and its eigenvalues are all unitary, and g is loxodromic otherwise. That classification, with these same definitions, extends to PSL$(n + 1, \mathbb{C})$ for all $n \geq 2$, see [15,16,24].

In this work we look at PSL$(3, \mathbb{C})$, the group of automorphisms of the projective plane $\mathbb{P}^2_{\mathbb{C}}$, and we classify its discrete subgroups that (besides the identity) have only parabolic elements. These are called *purely parabolic groups*. In order to describe our results we remark that every purely parabolic discrete group in PSL$(3, \mathbb{C})$ has a global fixed point $p \in \mathbb{P}^2_{\mathbb{C}}$ and therefore $\mathbb{P}^2_{\mathbb{C}} \setminus \{p\}$, one has a canonical holomorphic projection map π from $\mathbb{P}^2_{\mathbb{C}} \setminus \{p\}$ into $\ell \cong \mathbb{P}^1_{\mathbb{C}}$. This defines a group homomorphism:

$$\Pi = \Pi_{p,\ell,G} : \mathrm{PSL}(3, \mathbb{C}) \to Bihol(\ell) \cong \mathrm{PSL}(2, \mathbb{C}) \ ,$$
$$\Pi(g)(x) = \pi(g(x))$$

which is independent of ℓ up to conjugation. Its restriction to G is the control morphism of G and its image $\Pi(G)$ is the control group (see Definition 1.12 and [16]).

The various types of purely parabolic groups in PSL$(3, \mathbb{C})$ are fully described in Section 2, where we also describe their algebraic and dynamical properties. There are five such main families, these are:

- Elliptic groups. These are the only ones that are not conjugate to subgroups of the Heisenberg group Heis$(3, \mathbb{C})$ and they are subgroups of fundamental groups of elliptic surfaces (see [8]).
- Torus groups. These are subgroups of fundamental groups of complex tori:

$$\mathcal{T}(\mathfrak{L}) = \left\{ \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} : (a, b) \in \mathfrak{L} \right\} \ ,$$

where $\mathfrak{L}$ is an additive discrete subgroup of $\mathbb{C}^2$.

- Dual torus groups,

$$\mathcal{T}^*(\mathfrak{L}) = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : (a,b) \in \mathfrak{L} \right\} .$$

These split into three types: the first have Kulkarni limit set (see Definition 1.5) a complex projective line; the second have Kulkarni limit set a cone of projective lines over a circle, and the third type have all $\mathbb{P}^2_{\mathbb{C}}$ as Kulkarni limit set. From now on, for simplicity, we will say only limit set instead Kulkarni limit set, unless we specify otherwise.

- Inoue groups of three types. The first have limit set a cone of lines over a circle, the others have limit set all of $\mathbb{P}^2_{\mathbb{C}}$. The three types can be distinguished by the limit set and the control morphism (see Proposition 2.11).

- Kodaira groups $\mathcal{K}_0$, which are Abelian, and their extensions. There are five types of extensions $\mathcal{K}_i$, $i = 1, \cdots, 5$, which are purely parabolic and discrete. The first type $\mathcal{K}_1$ have limit set a projective line; the second type $\mathcal{K}_2$ have limit set a cone of projective lines over the circle. The remaining three types have limit set all of $\mathbb{P}^2_{\mathbb{C}}$ and they are distinguished by their control morphism (see details in Section 2).

In this work we prove:

Theorem 0.1. *Let $G \subset \mathrm{PSL}(3, \mathbb{C})$ be a purely parabolic discrete subgroup. Then:*

(1) *G is either virtually elliptic or virtually conjugate to a subgroup of the Heisenberg group $\mathrm{Heis}(3, \mathbb{C})$, which is itself purely parabolic.*

(2) *Its Kulkarni limit set Λ_{Kul} is either a line, a cone of lines over a circle, or the whole of $\mathbb{P}^2_{\mathbb{C}}$, and up to conjugation:*

   (a) *Λ_{Kul} is a line if, and only if, G is an elliptic group, a torus group, a dual torus group of type I, Abelian Kodaira group or a $\mathcal{K}_1$ group.*

   (b) *If Λ_{Kul} is a cone of lines over a circle, then G is either a dual torus group of type II, a Kleinian Inoue group, or a extended Kodaira group $\mathcal{K}_2$.*

   (c) *If $\Lambda_{Kul} = \mathbb{P}^2_{\mathbb{C}}$, then G is a dual Torus group of type III, a discrete non-Kleinian Inoue group, an extended Inoue group, or an extended Kodaira group $\mathcal{K}_i$ for $i = 3, 4, 5$.*

Concerning the dynamics we have:

Theorem 0.2. *Let $G \subset \mathrm{PSL}(3, \mathbb{C})$ be as in the theorem above and let Λ^*_{CoG} be the Conze-Guivarc'h limit set of the action on the dual $\check{\mathbb{P}}^2_{\mathbb{C}}$. Then:*

(1) *If Λ_{Kul} is a line and G is not a dual torus group, then Λ^*_{CoG} is the projective dual of Λ_{Kul} and it is the unique minimal set.*

(2) *If Λ_{Kul} is a cone of lines over a circle, then Λ^*_{CG} contains a projective real line and it is not a minimal set because there is always a global fixed point.*

(3) *If $\Lambda_{Kul} = \mathbb{P}^2_{\mathbb{C}}$, then: Λ^*_{CoG} contains a complex projective line and it is never a minimal set (as before, because there is a global fixed point).*

The theorem below is essential for our proof of Theorem 0.1.

Theorem 0.3. *Let $G \subset \mathrm{Heis}(3, \mathbb{C})$ be a discrete group, then G is purely parabolic and:*

(1) *There are $B_1, \ldots, B_n$ subgroups of G such that $G = Ker(\Pi|_G) \rtimes B_1 \rtimes \cdots \rtimes B_n$ and each B_i is isomorphic to $\mathbb{Z}^{k_i}$, for some $k_i \in \mathbb{N} \cup \{0\}$.*

(2) *$rank(Ker(\Pi|_G)) + \sum_{i=1}^{n} k_i \leq 6$.*

(3) *If G is complex Kleinian, then $rank(Ker(\Pi|_G)) + \sum_{i=1}^{n} k_i \leq 4$.*

We recall (see Definition 1.4) that G is complex Kleinian if it is discrete and there is a non-empty open invariant set where G acts properly discontinuously.

The proof of Theorem 0.3 uses in an essential way the obstructor dimension of a group G, introduced by Bestvina, Kapovich and Kleiner in [12]. This is a lower bound for the "action dimension" of G, and it is based on the classical van Kampen obstruction for embedding a simplicial complex into an Euclidean space [31]. Theorem 0.3 strengthens, for discrete groups in $\mathrm{Heis}(3, \mathbb{C})$, a Theorem of Bieri and Strebel [10], ensuring that every infinite, finitely presented solvable group is virtually an ascending HNN-extension of a finitely generated solvable group.

A difficulty one meets when working with the projective groups $\mathrm{PSL}(n+1, \mathbb{C})$, which are non-compact, is that one does not have the convergence property (cf. [13,21]). We overcome this problem by using repeatedly the space of pseudo-projective maps,

$$\mathrm{SP}(3, \mathbb{C}) = (M(3, \mathbb{C}) - \{\mathbf{0}\})/\mathbb{C}^*,$$

where $M(3, \mathbb{C})$ is the set of all 3×3 matrices with complex coefficients and $\mathbb{C}^*$ acts by the usual scalar multiplication. This was introduced in [16,17] and it provides a natural compactification of the projective group $\mathrm{PSL}(3, \mathbb{C})$.

We remark that every parabolic element in $\mathrm{PSL}(3, \mathbb{C})$ is conjugate to a parabolic element in $\mathrm{PU}(2, 1)$, the group of holomorphic isometries of the complex hyperbolic space (see [16,24]). Yet, the results in this paper show that there are plenty of purely parabolic groups in $\mathrm{PSL}(3, \mathbb{C})$ which are not conjugate to subgroups in $\mathrm{PU}(2, 1)$. Some examples are:

(1) All purely parabolic groups whose limit set Λ_{Kul} is not a single line.

(2) Unipotent Abelian complex Kleinian groups whose limit set is a single line and the rank is at least three.

It is possible to provide a full characterization of the purely parabolic groups in $\mathrm{PSL}(3,\mathbb{C})$ that are conjugate to groups in $\mathrm{PU}(2,1)$. This shall be done elsewhere.

A corollary of the results in this article is that the Kulkarni limit set of purely parabolic complex Kleinian groups in $\mathrm{PSL}(3,\mathbb{C})$ either consists of one line or it has infinitely many lines but only two in general position (a set of lines is said to be in general position if no three lines in the set are concurrent). This is essential for the classification of the complex Kleinian groups that are elementary, *i.e.*, they have either a point or a line with finite orbit. The complete classification of the elementary groups in $\mathrm{PSL}(3,\mathbb{C})$ is given in our forthcoming paper [6], in relation with a dictionary in complex dimension two, inspired by Sullivan's dictionary [7].

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1. Preliminaries

Let $\mathbb{P}^2_{\mathbb{C}} := (\mathbb{C}^3 \setminus \{\mathbf{0}\})/\mathbb{C}^*$ be the complex projective plane and $[\;] : \mathbb{C}^3 \setminus \{\mathbf{0}\} \to \mathbb{P}^2_{\mathbb{C}}$ the quotient map. A line in $\mathbb{P}^2_{\mathbb{C}}$ means the image under this projection of a complex linear subspace of dimension 2. Given $p, q \in \mathbb{P}^2_{\mathbb{C}}$ distinct points, there exists a unique complex line passing through p and q; such line is denoted by $\overleftrightarrow{p,q}$. The projective dual $\check{\mathbb{P}}^2_{\mathbb{C}}$ of $\mathbb{P}^2_{\mathbb{C}}$ is $Gr(\mathbb{P}^2_{\mathbb{C}})$ the Grassmannian of all complex lines in $\mathbb{P}^2_{\mathbb{C}}$ equipped with the topology of the Hausdorff convergence.

The following notion is used along the paper.

Definition 1.1. A pencil of lines in $\mathbb{P}^2_{\mathbb{C}}$ is a collection of lines passing through a common point.

Consider the usual action of $\mathbb{Z}_3$ on $\mathrm{SL}(3,\mathbb{C})$. Then $\mathrm{PSL}(3,\mathbb{C}) = \mathrm{SL}(3,\mathbb{C})/\mathbb{Z}_3$ is a Lie group whose elements are called projective transformations. We denote also by $[\;] : \mathrm{SL}(3,\mathbb{C}) \to \mathrm{PSL}(3,\mathbb{C})$ the quotient map. We denote by $\mathbf{g} = (g_{ij})$ the elements in $\mathrm{SL}(3,\mathbb{C})$. Given $g \in \mathrm{PSL}(3,\mathbb{C})$, we say that $\mathbf{g} \in \mathrm{SL}(3,\mathbb{C})$ is a lift of g if $[\mathbf{g}] = g$. Then $\mathrm{PSL}(3,\mathbb{C})$ acts transitively, effectively and by biholomorphisms on $\mathbb{P}^2_{\mathbb{C}}$ by $[\mathbf{g}]([w]) = [\mathbf{g}(w)]$, where $w \in \mathbb{C}^3 \setminus \{\mathbf{0}\}$ and $\mathbf{g} \in \mathrm{SL}(3,\mathbb{C})$. Recall (cf. [16, Chapter 4]):

Definition 1.2. Let $g \in \mathrm{PSL}(3,\mathbb{C})$ and $\mathbf{g}$ a lift to $\mathrm{SL}(3,\mathbb{C})$. Then g is:

- elliptic if $\mathbf{g}$ is diagonalizable with unitary eigenvalues;
- parabolic if $\mathbf{g}$ is non-diagonalizable with unitary eigenvalues;
- loxodromic if $\mathbf{g}$ has some non-unitary eigenvalue.

Now let $M(3,\mathbb{C})$ be the set of all 3×3 matrices with complex coefficients. Define the space of pseudo-projective maps by: $\mathrm{SP}(3,\mathbb{C}) = (M(3,\mathbb{C}) - \{\mathbf{0}\})/\mathbb{C}^*$, where $\mathbb{C}^*$

acts on $M(3,\mathbb{C}) - \{\mathbf{0}\}$ by the usual scalar multiplication. We have the quotient map $[\,] : M(3,\mathbb{C}) \setminus \{\mathbf{0}\} \to \mathrm{SP}(3,\mathbb{C})$. Given $P \in \mathrm{SP}(3,\mathbb{C})$ we define its kernel by:

$$Ker(P) = [Ker(\mathbf{P}) \setminus \{\mathbf{0}\}],$$

where $\mathbf{P} \in M(3,\mathbb{C})$ is a lift of P. Clearly $\mathrm{PSL}(3,\mathbb{C}) \subset \mathrm{SP}(3,\mathbb{C})$ and an element P in $\mathrm{SP}(3,\mathbb{C})$ is in $\mathrm{PSL}(3,\mathbb{C})$ if and only if $Ker(P) = \emptyset$. Notice that $\mathrm{SP}(3,\mathbb{C})$ is a manifold naturally diffeomorphic to $\mathbb{P}^8_{\mathbb{C}}$, so it is compact.

Recall that given a discrete group G in $\mathrm{PSL}(3,\mathbb{C})$, its equicontinuity set $\mathrm{Eq}(G)$ is the largest open set on which the G-action forms a normal family.

Theorem 1.3 *(See Proposition 2.5 in [15]). Let $G \subset \mathrm{PSL}(3,\mathbb{C})$ be a discrete group. Then G acts properly discontinuously on $\mathrm{Eq}(G)$ and one has:*

$$\mathrm{Eq}(G) = \mathbb{P}^2_{\mathbb{C}} \setminus \overline{\bigcup Ker(P)}\,,$$

where the union runs over the kernels of all $P \in \mathrm{SP}(3,\mathbb{C}) \setminus \mathrm{PSL}(3,\mathbb{C})$ satisfying that there exists a sequence $(g_n) \subset G$ that converges to P.

Now let $G \subset \mathrm{PSL}(3,\mathbb{C})$ be a discrete group and let Ω be a non-empty G-invariant set, *i.e.*, $G\Omega = \Omega$. We say that G *acts properly discontinuously* on Ω if for each compact set $K \subset \Omega$ the set $\{g \in G \,|\, g(K) \cap K\}$ is finite.

The following notion was introduced in [28]:

Definition 1.4. G is *complex Kleinian* if there exists a non-empty open G-invariant set in $\mathbb{P}^2_{\mathbb{C}}$ where G acts properly discontinuously.

Definition 1.5. Let $G \subset \mathrm{PSL}(2,\mathbb{C})$ be a discrete group. Its *Kulkarni limit set* is [22]:

$$\Lambda_{Kul}(G) = L_0(G) \cup L_1(G) \cup L_2(G)\,,$$

where $L_0(G)$ is the closure of the points in $\mathbb{P}^2_{\mathbb{C}}$ with infinite isotropy group, $L_1(G)$ is the closure of the set of accumulation points of the orbits Gz where z runs over $\mathbb{P}^2_{\mathbb{C}} \setminus L_0(G)$, and $L_2(G)$ is the closure of the set of accumulation points of orbits GK where K runs over all compact sets in $\mathbb{P}^2_{\mathbb{C}} - (L_0(G) \cup L_1(G))$. The *Kulkarni region of discontinuity (or the ordinary set)* of G is:

$$\Omega_{Kul}(G) = \mathbb{P}^2_{\mathbb{C}} \setminus \Lambda_{Kul}(G).$$

Proposition 1.6. *Let G be a complex Kleinian group. Then:*

(1) [See [22]] *The sets $\Lambda_{Kul}(G)$, $L_0(G)$, $L_1(G)$, $L_2(G)$ are G-invariant closed sets.*
(2) (See [22]) *The group G acts properly discontinuously on $\Omega_{Kul}(G)$.*

(3) (See [24] or [16, Proposition 3.3.6]) *Let $\mathcal{C} \subset \mathbb{P}^2_{\mathbb{C}}$ be a closed G-invariant set such that for every compact set $K \subset \mathbb{P}^2_{\mathbb{C}} - \mathcal{C}$, the set of cluster points of GK is contained in $(L_0(G) \cup L_1(G)) \cap \mathcal{C}$, then $\Lambda_{Kul}(G) \subset \mathcal{C}$.*

(4) (See [15, Corollary 2.6]) *The equicontinuity set of G is contained in $\Omega_{Kul}(G)$.*

(5) (See [5, Proposition 3.6]) *If $G_0 \subset G$ is a subgroup with finite index, then*

$$\Lambda_{Kul}(G) = \Lambda_{Kul}(G_0).$$

(6) (See [16, Proposition 3.3.4]) *The set $\Lambda_{Kul}(G)$ contains at least one complex line.*

There is also the Conze-Guivarc'h limit set, see [19]. For this we need the following generalization introduced in [9].

Definition 1.7. Let $G \subset \mathrm{PSL}(3, \mathbb{C})$ be a discrete group acting on $\check{\mathbb{P}}^2_{\mathbb{C}}$. We define the Conze-Guivarc'h limit set of G, denoted $\Lambda_{CoG}(G)$, as the closure of the set of points $q \in \check{\mathbb{P}}^2_{\mathbb{C}}$ for which there exist an open subset $U \subset \check{\mathbb{P}}^2_{\mathbb{C}}$ and a sequence $(g_n) \subset G$, $g_n = g_m$ if $n = m$, such that for every $p \in U$

$$\lim_{n \to \infty} g_n(p) = q.$$

A couple of examples will picture the previous concept. We need:

Definition 1.8. A matrix $g \in \mathrm{GL}(3, \mathbb{C})$ is *proximal* if it has an eigenvalue $\lambda_0 \in \mathbb{C}$ such that $|\lambda_0| > |\lambda|$ for all other eigenvalues λ of g. For such a g, an eigenvector $v_0 \in \mathbb{C}^3$ corresponding to the eigenvalue λ_0 is a *dominant eigenvector* of g. We say that $g \in \mathrm{PSL}(3, \mathbb{C})$ is *proximal* if it has a lift $\widetilde{g} \in \mathrm{SL}(3, \mathbb{C})$ which is proximal; and $v \in \mathbb{P}^2_{\mathbb{C}}$ is *dominant* for g if there is a lift $\tilde{v} \in \mathbb{C}^3$ of v which is dominant for $\widetilde{\gamma}$.

We remark that by [24], every strongly loxodromic element in $\mathrm{PSL}(3, \mathbb{C})$ is proximal, and all loxodromic elements in $\mathrm{PU}(2, 1)$ are strongly loxodromic.

Example 1.9. [Complex hyperbolic groups] If $G \subset \mathrm{PU}(2, 1)$ is a non-elementary discrete subgroup, then $\Lambda_{CoG}(\Gamma)$ coincides with the Chen-Greenberg limit set [18] of G, $\Lambda_{CG}(\Gamma)$, which is the closure of the orbits of points in the complex hyperbolic space $\partial \mathbb{H}^2_{\mathbb{C}}$. This follows because loxodromic elements in $\mathrm{PU}(2, 1)$ are proximal and their attracting fixed points in $\partial \mathbb{H}^2_{\mathbb{C}}$ correspond to dominant vectors.

Example 1.10. [Veronese groups] Given a non-elementary discrete subgroup $G \subset \mathrm{PSL}(2, \mathbb{C})$, let $\iota : \mathrm{PSL}(2, \mathbb{C}) \to \mathrm{PSL}(3, \mathbb{C})$ be the canonical irreducible representation and $\psi : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ the Veronese embedding. A simple computation shows that ι carries loxodromic elements in $\mathrm{PSL}(2, \mathbb{C})$ into strongly loxodromic elements in $\mathrm{PSL}(3, \mathbb{C})$. This implies $\Lambda_{CoG}(\iota(G)) = \psi(\Lambda_{ChG}(G))$, see [14, Theorem 2.10].

Recall from [19] that a *strongly irreducible group* in $\mathrm{PSL}(3,\mathbb{C})$ is a group whose action on $\mathbb{P}^2_{\mathbb{C}}$ does not have points or lines with finite orbit.

Theorem 1.11 *(See [19] and Corollary 3 in [9]). Let $G \subset \mathrm{PSL}(3,\mathbb{C})$ be a strongly irreducible group, then:*

(1) *The limit set $\Lambda_{CoG}(G)$ is non-empty and is the unique minimal set for the action of G on $\mathbb{P}^2_{\mathbb{C}}$.*
(2) *The closure of the dominant points of proximal elements in G coincides with $\Lambda_{CoG}(G)$.*

Now recall [16]:

Definition 1.12. Let G be a discrete group in $\mathrm{PSL}(3,\mathbb{C})$. We say that G is *weakly-controllable* if it acts with a fixed point p in $\mathbb{P}^2_{\mathbb{C}}$. In this case a choice of a line $\mathcal{L}$ in $\mathbb{P}^2_{\mathbb{C}} \setminus \{p\}$ determines a projection map $\mathbb{P}^2_{\mathbb{C}} \setminus \{p\} \to \mathcal{L}$ and a group morphism $\Pi : G \to \mathrm{PSL}(2,\mathbb{C})$ called the *control morphism* of G; its image $\Pi(G) \subset \mathrm{PSL}(2,\mathbb{C})$ is the *control group*. These are well defined and independent of $\mathcal{L}$ up to an automorphism of $\mathrm{PSL}(2,\mathbb{C})$.

Theorem 1.13 *(See Theorem 5.8.2 in [16]). Let $G \subset \mathrm{PSL}(3,\mathbb{C})$ be discrete and weakly-controllable, with $p \in \mathbb{P}^2_{\mathbb{C}}$ a G-invariant point and $\ell \subset \mathbb{P}^2_{\mathbb{C}}$ a complex line not containing p. Let $\Pi_{p,\ell} = \Pi$ be a projection map defined as above. If $Ker(\Pi|_G)$ is finite and $\Pi(G) \subset \mathrm{Aut}(\ell) \cong \mathrm{PSL}(2,\mathbb{C})$ is discrete, then G acts properly discontinuously on*

$$\Omega = \left(\bigcup_{z \in \Omega(\Pi(G))} \overleftrightarrow{p,z} \right) - \{p\} \, ,$$

where the union runs over all points in ℓ where the action of $\Pi(G)$ is discontinuous.

The following is an improvement of the λ-Lemma in [24] that we use in the sequel. This is inspired by the classical λ-Lemma of Palis and De Melo [25].

Lemma 1.14 *(See Section 2 in* Heis *[15]). Let G be a discrete group and let $(g_n) \subset G$ be a sequence of distinct elements, then there exist a subsequence $(h_n) \subset (g_n)$ and pseudo projective maps $P, Q \in SP(3,\mathbb{C})$ satisfying:*

(1) $\quad h_n \xrightarrow[m \to \infty]{} P \quad and \quad h_n^{-1} \xrightarrow[m \to \infty]{} Q \ .$

(2)

$$
\begin{aligned}
&Im(P) \subset Ker(Q)\,, \\
&Im(Q) \subset Ker(P)\,, \\
&dim(Im(P)) + dim(Ker(P)) = 1\,, \\
&dim(Im(Q)) + dim(Ker(Q)) = 1\,.
\end{aligned}
$$

(3) *For every point $x \in Ker(P)$ we get*

$$
Ker(Q) = \bigcup_{x_n \to x} \{accumulation\ points\ of\ (h_n(x_n))\}.
$$

(4) *If $\Omega \subset \mathbb{P}^2_{\mathbb{C}}$ is an open set on which G acts properly discontinuously, then either $Ker(P) \subset \mathbb{P}^2_{\mathbb{C}} - \Omega$ or $Ker(Q) \subset \mathbb{P}^2_{\mathbb{C}} - \Omega$.*

We use in the sequel the obstructor dimension of a group G, introduced by Bestvina, Kapovich and Kleiner in [12]. We refer to [12, Definition 4].

Definition 1.15. Fix a non-negative integer m. A finite simplicial complex K of dimension $\leq m$ is an m-obstructor complex if the following holds:

(1) There is a collection

$$
\Sigma = \{(\sigma_i, \tau_i)_{i=1}^k\}
$$

of unordered pairs of disjoint simplices of K with $dim\,\sigma_i + dim\,\tau_i = m$ that determine an m-cycle (over $\mathbb{Z}_2$) in

$$
\{\sigma \times \tau \subset K \times K | \sigma \cap \tau = \emptyset\} / \mathbb{Z}_2
$$

where $\mathbb{Z}_2$ acts by $(x, y) \mapsto (y, x)$.
(2) For some (any) general position map $f : K \to \mathbb{R}^m$ the (finite) number $\Sigma_{i=1}^k |f(\sigma_i) \cap f(\tau_i)|$ is odd.
(3) For every m-simplex $\sigma \in K$ the number of vertices v such that the unordered pair $\{\sigma, \tau\}$ is in Σ is even.

Definition 1.16. The obstructor dimension $obdim(G)$ is defined to be 0 for finite groups, 1 for 2-ended groups (see [20, Section 9.1] for a definition of the ends of a group). Otherwise $obdim(G)$ is $m + 2$ where m is the largest integer such that for some m-obstructor complex K and some triangulation of the open cone $cone(K)$ there exists a proper map $f : cone(K)^{(0)} \to G$ satisfying:

(1) for disjoint simplices σ, τ in K and every $D > 0$ there are compact sets $C_1 \subset cone(\sigma)$, $C_2 \subset cone(\tau)$ such that $f(cone(\sigma) - C_1)$ and $f(cone(\sigma) - C_2)$ are $> D$ apart.

(2) there is a uniform upper bound on the distance between the images of adjacent vertices in $cone(K)^{(0)}$

We use the following theorems; see [11,12] for the corresponding proofs:

Theorem 1.17 *(See Theorem 1 in [12]). If $obdim(G) \geq m$, then G can not act properly discontinuously on a contractible manifold of dimension $< m$.*

Theorem 1.18 *(See Corollary 2.7 in [12]). If $G = H \rtimes Q$ with H and Q finitely generated and H weakly convex, then $obdim(G) \geq obdim(H) + obdim(Q)$.*

Theorem 1.19 *(See Corollary 2.2 in [11]). Let G be a lattice in a simply connected nilpotent Lie group N, then $obdim(G) = dim(N)$, in particular $obdim(\mathbb{Z}^n) = n$.*

2. The families of purely parabolic discrete groups

There is a partition of the purely parabolic discrete groups in $\mathrm{PSL}(3,\mathbb{C})$ into five families: elliptic, torus, dual torus, Inoue and Kodaira groups. We now define these families and discuss their algebraic structure and dynamical properties. We find that some of these families naturally split into subfamilies according to the topology of their limit set and the structure of the control group.

Note that the simplest purely parabolic groups are cyclic, generated by a parabolic element; there are three types of such elements in $\mathrm{PSL}(3,\mathbb{C})$, described by the Jordan normal form of their lifts to $\mathrm{SL}(3,\mathbb{C})$. These are:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}, |\lambda| = 1,\, \lambda \neq 1.$$

The first two of these are unipotent; the third type is called ellipto-parabolic: it is rational if λ is a root of unity or irrational otherwise (see [16, Chapter 4] for details). Each of these belongs to a different type of the families we describe below. The first type generates torus groups, the second generates Abelian Kodaira groups and the ellipto-parabolic elements generate elliptic groups.

We remark that all the groups we present in this section are upper triangular. Hence they fix the point $e_1 \in \mathbb{P}^2_\mathbb{C}$, so they are weakly-controllable. Given one of these groups G, we let $\Pi : G \to \mathrm{PSL}(2,\mathbb{C})$ be its control morphism (Definition 1.12). Notice that except for the elliptic groups, all others are subgroups of the Heisenberg group.

Throughout this section we denote by W an additive subgroup of $\mathbb{C}$, by $\mathfrak{L}$ and additive subgroup of $\mathbb{C}^2$ and by $\mathcal{M}$ an additive subgroup of $\mathbb{R}$.

2.1. Elliptic groups

Let $W \subset \mathbb{C}$ be an additive discrete subgroup and consider a group morphism $\mu : W \to \mathbb{S}^1$. Define:

$$\mathrm{Ell}(W, \mu) = \left\{ \begin{bmatrix} \mu(w) & \mu(w)w & 0 \\ 0 & \mu(w) & 0 \\ 0 & 0 & \mu(w)^{-2} \end{bmatrix} : w \in W \right\}.$$

Lemma 2.1. *Elliptic groups act with two fixed points, $\{e_1\}$ and $\{e_3\}$. The control group with respect to $\{e_3\}$ is W and the kernel of Π is trivial, while the control group with respect to $\{e_1\}$ is the image of W under μ. Also:*

- *The Kulkarni limit set is a line.*
- *The equicontinuity set coincides with Kulkarni's discontinuity region and is the largest open set on which the group acts properly discontinuously.*
- *The Conze-Guivarc'h limit set of the action on the dual $\check{\mathbb{P}}_{\mathbb{C}}^2$ is the dual point of the Kulkarni limit set, but it is not the only minimal set.*

The kernel may or may not be trivial. If the kernel is not trivial, then it is a proper subgroup of W isomorphic to $\mathbb{Z}$.

Proof. The proof of the statements about the control groups are straightforward from the definition. From Theorem 1.13 we have that the Kulkarni limit set is the line $\overleftrightarrow{e_1, e_2}$ and the Kulkarni's discontinuity region is the largest open set on which $\mathrm{Ell}(W, \mu)$ acts properly discontinuously. Let us prove that Ω_{Kul} coincides with the equicontinuity set. Let $(g_n) \subset \mathrm{Ell}(W, \mu)$ be a sequence of distinct elements. Then (g_n) can be written as:

$$g_n = \begin{bmatrix} 1 & a_n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu^{-3}(a_n) \end{bmatrix},$$

for some sequence (a_n) in W with $|a_n|$ converging to ∞. Hence there exists $a \in \mathbb{C}^*$ such that:

$$g_n \xrightarrow[n \to \infty]{} \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ; \text{ and } g_n^* \xrightarrow[n \to \infty]{} \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We get $Eq(\mathrm{Ell}(W, \mu)) = \mathbb{C}^2 = \Omega_{Kul}(\mathrm{Ell}(W, \mu))$ and $\Lambda_{CoG}(\mathrm{Ell}(W, \mu)) = \{e_2\}$ is a minimal set. This also proves the statements about the Conze-Guivarc'h limit set. Notice that $\{e_3\}$ also is a minimal set for the dual action, so there is more than one minimal set. $\square$

2.2. Torus groups

These are of the form:

$$T(\mathfrak{L}) = \left\{ \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} : (a,b) \in \mathfrak{L} \right\},$$

where $\mathfrak{L}$ is an additive discrete subgroup of $\mathbb{C}^2$.

Lemma 2.2. *The control group $\Pi(G)$ is an additive subgroup of $\mathbb{C}$ that may or may not be discrete, and:*

- *The Kulkarni limit set Λ_{Kul} is a line.*
- *The equicontinuity set coincides with the Kulkarni discontinuity region and it is the largest open set where the group acts properly discontinuously.*
- *The Conze-Guivarc'h limit set $\Lambda^*_{CoG}(T(W))$ of the action on the dual projective plane $\check{\mathbb{P}}^2_{\mathbb{C}}$ is a single point, the projective dual of the unique line in Λ_{Kul}, and it is the only minimal set for the action on $\check{\mathbb{P}}^2_{\mathbb{C}}$.*

The kernel of the control morphism Π may or may not be trivial.

Proof. The first and second statements follow from [16, 3.4.2]; in fact Λ_{Kul} is the line $\overleftrightarrow{e_1, e_2}$. It remains to prove the statements about $\Lambda^*_{CoG}(T(\mathfrak{L}))$. Let $(g_n) \subset T(\mathfrak{L})$ be a sequence of distinct elements. Choose a sequence $(a_n, b_n) \in \mathfrak{L}$ such that $|a_n| + |b_n| \xrightarrow[n\to\infty]{} \infty$ and set:

$$g_n = \begin{bmatrix} 1 & 0 & a_n \\ 0 & 1 & b_n \\ 0 & 0 & 1 \end{bmatrix}.$$

We can assume that (taking a subsequence if necessary) there exist $a, b \in \mathbb{C}$ so that $|a| + |b| \neq 0$ and

$$g_n \xrightarrow[n\to\infty]{} \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence

$$g_n^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a_n & -b_n & 1 \end{bmatrix}$$

converges to $\{e_3\}$. Thus $\Lambda^*_{CoG}(T(L)) = \{e_3\}$ and this set is minimal. $\square$

2.3. Dual Torus groups

This family actually splits in three classes, depending on the limit set. To explain this trichotomy, let us consider first the following lemma. We recall that the rank of a group is the smallest number of elements that generate it.

Lemma 2.3. *Let $\mathcal{L}$ be an additive discrete subgroup of $\mathbb{C}^2$ and consider the natural projection $[\,] : \mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1_{\mathbb{C}}$. Then the closure of $[\mathcal{L} \setminus \{0\}]$ is either a point, a real projective line or the whole of $\mathbb{P}^1_{\mathbb{C}}$, and one has:*

- *The closure of $[\mathcal{L} \setminus \{0\}]$ is a point if and only if $\mathcal{L}$ has rank 1 or it has rank 2 and it is generated by two $\mathbb{C}$-linearly dependent elements.*
- *The closure of $[\mathcal{L} \setminus \{0\}]$ is a real projective line if and only if $\mathcal{L}$ has rank 2 and it is generated by two $\mathbb{C}$-linearly independent elements.*
- *The closure of $[\mathcal{L} \setminus \{0\}] = \mathbb{P}^1_{\mathbb{C}}$ if and only if $\mathcal{L}$ has rank at least 3.*

Proof. If $\mathcal{L}$ has rank 1, then trivially $[\mathcal{L} \setminus \{0\}]$ is a single point. If $\mathcal{L}$ has rank two and is generated by two $\mathbb{C}$-linearly dependent vectors, then $[\mathcal{L} \setminus \{0\}]$ is trivially a single point. If $\mathcal{L}$ has rank two and is generated by two $\mathbb{C}$-linearly independent vectors, then we can assume that $\mathcal{L} = \mathbb{Z} \oplus \mathbb{Z}$, therefore $[\mathcal{L} \setminus \{0\}] = \{[1, m/n] : m, n \in \mathbb{Z}\}$ which is a dense set in a real projective line in $\mathbb{P}^1_{\mathbb{C}}$. Finally, if $\mathcal{L}$ has rank at least 3, then we can pick up two elements in $\mathcal{L}$ which are $\mathbb{C}$-linearly independent. Moreover we can assume that $(1, 0)$ and $(0, 1)$ are such elements, let $p = (w_1, w_2)$ be the other point in $\mathcal{L}$, then

$$\overline{[\Pi \setminus \{0\}]} = \overline{\{[k + nw_1 : l + nw_2] : k, l, n \in \mathbb{Z}\}} = \overline{\{[r + sw_1 : t + sw_2] : r, s, t \in \mathbb{R}\}}\,.$$

Now, if $z \in \mathbb{C}$ satisfies $Im(z) \neq 0$ and we consider

$$s_0 = 1, \ r_0 = \frac{Im(w_2) - Re(z)Im(w_1)}{Im(z)} - Re(w_1), \ t_0 = z(r + w_1) - w_2\,,$$

then a straightforward computation shows that $[1 : z] = [r_0 + s_0 w_1 : t_0 + s_0 w_2]$, which concludes the proof. $\quad\square$

Definition 2.4. A *dual torus group* in $\mathrm{PSL}(3, \mathbb{C})$ is a group of the form

$$\mathcal{T}^*(\mathfrak{L}) = \left\{ g_{(a,b)} = \begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : (a, b) \in \mathfrak{L} \right\},$$

where $\mathfrak{L} \subset \mathbb{C}^2$ is an additive discrete group. *The torus group is of type I if its Kulkarni limit set Λ_{Kul} is a line; it is of type II if Λ_{Kul} is a cone of lines over a circle, and it is of type III if $\Lambda_{\mathrm{Kul}} = \mathbb{P}^2_{\mathbb{C}}$.*

Lemma 2.5. *All dual torus groups are discrete, with trivial control group, and:*

(1) *The group is:*
   (a) *Type I if and only if $\mathfrak{L}$ either has rank 1 or it is generated by two $\mathbb{C}$-linearly dependent elements;*
   (b) *Type II if and only if $\mathfrak{L}$ has rank 2 and it is generated by two $\mathbb{C}$-linearly independent elements;*
   (c) *Type III if and only if $\mathfrak{L}$ has rank at least 3.*
(2) *If it is of type I or II, then:*
   (a) *Its Kulkarni discontinuity set is the largest open set where the action is properly discontinuously and it coincides with the equicontinuity set.*
   (b) *The Conze-Guivarc'h limit set of the dual action is the projective dual of Λ_{Kul}; it is either a point if the group is of type I or a real projective line if it is of type II, and it is a minimal set whenever it is a single point.*
(3) *If the group is of type III, then:*
   (a) *The sets Ω_{Kul} and Eq are both empty and there is no non-empty open invariant set of $\mathbb{P}^2_{\mathbb{C}}$ where the group acts properly discontinuously.*
   (b) *The Conze-Guivarc'h is the projective dual of Λ_{Kul}, but this is not a minimal set since there is a global fixed point.*

Proof. It is not hard to check that these groups are discrete and they have trivial control group. The rest of the proof follows from Lemma 2.3. □

2.4. The Inoue groups

There are three classes of groups in this family.

a) *Inoue Kleinian* groups, or just Inoue groups. These are proper subgroups of fundamental groups of Inoue surfaces. The limit set is a cone of lines over a circle.

b) *Inoue non-Kleinian* groups. These are Inoue groups in the sense of Definition 2.6 whose limit set is all $\mathbb{P}^2_{\mathbb{C}}$.

c) *Extended Inoue* groups. These are finite extensions of Inoue groups whose limit set is all of $\mathbb{P}^2_{\mathbb{C}}$, hence they are not Kleinian.

2.4.1. a) Inoue groups

Let $\mathfrak{L} \subset \mathbb{C}^2$ be an additive discrete subgroup, let $x, y, z \in \mathbb{C}$ and set

$$\gamma_1 = \gamma_1(x, y, z) := \begin{bmatrix} 1 & x+z & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \ , \quad \mathcal{I} = \mathcal{I}(u, v) := \left\langle \begin{bmatrix} 1 & u & v \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ , \ (u, v) \in \mathfrak{L} \right\rangle .$$

Notice $\mathcal{I}$ is a dual torus group.

Definition 2.6. An Inoue group is a discrete subgroup of $\mathrm{PSL}(3, \mathbb{C})$ which is an extension $G = \langle \mathcal{I}, \gamma_1 \rangle$ where the dual torus group $\mathcal{I}$ is of type II. This kind of groups splits into two classes: Kleinian, *i.e.* subgroups with non-empty discontinuity region, and non-Kleinian.

Theorem 2.7. *A group Γ is Inoue if and only if there exists a dual torus group $\widetilde{I}$ such that:*

$$\Gamma = \{\widetilde{I}\,\gamma_1^k \,|\, k \in \mathbb{Z}\} \,.$$

These groups are non-Abelian semi-direct products $\mathbb{Z}^2 \rtimes \mathbb{Z}$. They are weakly-controllable with control group $\mathbb{Z}$ and kernel (of the control morphism) $\mathbb{Z} \oplus \mathbb{Z}$. Moreover:

(1) *The group is Kleinian if and only if it is of the form:*

$$\mathrm{Ino}(x,y,p,q,r) = \left\{ \begin{bmatrix} 1 & k + \frac{lp}{q} + mx & \frac{l}{r} + m\left(k + \frac{lp}{q}\right) + \binom{m}{2}x + my \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix} : k,l,m \in \mathbb{Z} \right\}$$

where $x, y \in \mathbb{C}$ and $p, q, r \in \mathbb{Z}$ are such that p, q are co-primes and q^2 divides r.

(2) *If the group is Kleinian, then:*

- *The Kulkarni limit set is a cone of lines over a real projective space:*

$$\Lambda_{Kul} = \overleftrightarrow{e_1, e_2} \cup \bigcup_{s \in \mathbb{R}} \overleftrightarrow{e_1, [0 : 1 : s]}.$$

- *The Kulkarni discontinuity set coincides with the equicontinuity set and it is the largest open set on which the group acts properly discontinuously. These sets are biholomorphic to $\mathbb{C} \times (\mathbb{H}^+ \cup \mathbb{H}^-)$ where $\mathbb{H}^\pm$ are the open half planes in $\mathbb{C}$.*

- *The Conze-Guivarc'h limit set for $\mathrm{Ino}(x,y,p,q,r)$ is a real projective line, and it is not minimal.*

Proof. Set $\widetilde{\mathcal{I}} = \{g \in \Gamma : g \in Ker(\Pi)\}$, then $\Gamma = \{h\gamma_1^k : k \in \mathbb{Z}, h \in \widetilde{\mathcal{I}}\}$ and $\widetilde{\mathcal{I}}$ is a dual torus groups, proving the first statement. On the other hand, it is clear that $\mathrm{Ino}(x,y,p,q,r)$ is a discrete group. Set:

$$g(k,l,m) = \begin{bmatrix} 1 & k + lc + mx & ld + m(k + lc) + \binom{m}{2}x + my \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $k, l \in \mathbb{Z}$, then a straightforward computation shows that the fixed point set is:

$$Fix(g(k,l,0)) = \overleftrightarrow{e_1, [0 : -ld : k + lc]}.$$

Hence, letting L_0 be as in Definition 1.5 we get:

$$\overrightarrow{e_1, e_3} \cup \bigcup_{s \in \mathbb{R}} \overleftarrow{e_1, [0:1:s]} \subset L_0(\text{Ino}(x, y, p, q, r)).$$

Finally, let us show that

$$\mathbb{P}_{\mathbb{C}}^2 - Eq(\text{Ino}(x, y, p, q, r)) = \overrightarrow{e_1, e_3} \cup \bigcup_{s \in \mathbb{R}} \overleftarrow{e_1, [0:1:s]}.$$

Let $(g_m)_{m \in \mathbb{N}} \subset \text{Ino}(x, y, p, q, r)$ be a sequence of distinct elements, then there exists a sequence $u_m = (k_m, l_m, n_m) \in \mathbb{Z}^3$ of distinct elements such that:

$$g_m = \begin{bmatrix} 1 & k_m + l_m c + n_m x & l_m d + n_m (k_m + l_m c) + \binom{n_m}{2} x + n_m y \\ 0 & 1 & n_m \\ 0 & 0 & 1 \end{bmatrix}.$$

Since G_w is discrete we get $r_m = \max\{|k_m|, |l_m|, |n_m|\} \xrightarrow[m \to \infty]{} \infty$. Now we can assume that there exists $u = (x, y, z) \in \mathbb{R}^3 - \{\mathbf{0}\}$ such that $r_m^{-1} u_m \xrightarrow[m \to \infty]{} u$, thus

$$g_m \xrightarrow[m \to \infty]{} P = \begin{bmatrix} 0 & k_0 + l_0 c + n_0 x & l_0 d + n_0 (k_0 + l_0 c) + \binom{n_0}{2} x + n_0 y \\ 0 & 0 & n_0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$Ker(P) = \begin{cases} \overleftrightarrow{e_1, e_2} & \text{if } k_0 + l_0 c + n_0 x = 0 \\ \overleftarrow{e_1, [0:-l_0 d : k_0 + l_0 c]} & \text{if } k_0 + l_0 c \neq 0, n_0 = 0 \\ e_1 & \text{in other case} \end{cases}$$

This last convergence implies that Λ_{CoG}^* is a real projective line. This set is not minimal because it has a global fixed point. $\square$

2.4.2. b) Extended Inoue groups

These are discrete extensions of Inoue groups. We use the following normal forms.

$$\mathfrak{g} = \begin{bmatrix} 1 & 1 & s \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} ; s \in \mathbb{C} . \tag{2.1}$$

Definition 2.8. An *extended Inoue group* is a discrete group $\widetilde{\text{Ino}}(\mathfrak{L}, x, y, z)$ generated by matrices with normal forms $\mathfrak{g}, \gamma_1$ and the group $\mathcal{I}$.

We have:

Lemma 2.9. *Up to conjugation, every extended Inoue group is of type:*

$$\widetilde{\mathrm{Ino}}(\mathfrak{L}, x, y, z) \;=\; \left\{ h \cdot \mathfrak{g}^k \cdot \gamma_1^m \,, k, m \in \mathbb{Z}, h \in \mathcal{I} \right\},$$

with $a, b, c, x \in \mathbb{C}$, $k, m \in \mathbb{Z}$ and $(u, v) \in \mathfrak{L}$ satisfying that if we let π_1, π_2 be the coordinate functions in $\mathbb{C}^2$, then:

i) $(0, x - z), (0, \pi_1(\mathfrak{L})), (0, z \cdot \pi_2(\mathfrak{L}))$ are in $\mathfrak{L}$.
ii) $\mathfrak{L}$ has rank at least 3; and
iii) either $z \notin \mathbb{R}$ or $x = y = z = 0$; hence the control group has rank 2.

The proof follows from Proposition 5.21.

Corollary 2.10. *The extended Inoue groups have an infinite discrete control group. There is not an open invariant set of $\mathbb{P}^2_{\mathbb{C}}$ where the group acts properly discontinuously; the sets Ω_{Kul} and Eq are both empty, the Kulkarni limit set is all of $\mathbb{P}^2_{\mathbb{C}}$ and the Conze-Guivarc'h limit set contains at least a complex projective line.*

Proof. By Lemma 2.9 the group $\mathcal{I}$ in 2.4.1 is a dual torus group with rank at least three. By Lemma 2.5 we deduce $L_0(\mathcal{I}) = \mathbb{P}^2_{\mathbb{C}}$, where L_0 is the first set in Kulkarni's limit set, so there is no open set on which an extended Inoue group acts properly discontinuously. Finally, we remark that by the last statement in the lemma above, the control group is the additive group spanned by z and 1, so by 2.9 we conclude that the control group has rank two and it is discrete. Note that the kernel of the control morphism is a dual torus group of type III, so Eq and Ω_{Kul} are both empty, the Kulkarni limit set is all of $\mathbb{P}^2_{\mathbb{C}}$ and the Conze-Guivarc'h limit set contains at least one complex projective line. □

The following is an immediate consequence of Theorem 2.7 and Lemma 2.9:

Proposition 2.11.

(1) *The Inoue Kleinian groups have limit set a cone of lines over a circle, the kernel of the control morphism is $\mathbb{Z} \oplus \mathbb{Z}$ and the control group is $\mathbb{Z}$.*
(2) *The Inoue non-Kleinian groups have limit set all of $\mathbb{P}^2_{\mathbb{C}}$, the kernel of the control morphism is $\mathbb{Z} \oplus \mathbb{Z}$ and the control group is $\mathbb{Z}$.*
(3) *The extended Inoue groups are finite extensions of Inoue groups (Kleinian or not). They have limit set all of $\mathbb{P}^2_{\mathbb{C}}$, the kernel of the control morphism is $\mathbb{Z}^k$ for some $k \geq 3$, and the control group is $\mathbb{Z} \oplus \mathbb{Z}$.*

2.5. Kodaira groups

There are:
a) Abelian Kodaira groups, $\mathcal{K}_0$.

b) Extended Kodaira groups $\mathcal{K}_i$, $i = 1, \ldots, 5$. These are all non-Abelian; they are finite extensions of Abelian Kodaira groups and they split into six types according to their limit set and the control group. These are:

(1) Groups $\mathcal{K}_i$, $i = 1, 2, 3$. These three classes are constructed in a similar way (see Lemma 2.16 below).
   - The groups $\mathcal{K}_1$ have limit set a complex projective line and discrete control group.
   - The groups $\mathcal{K}_2$ have limit set a cone of lines over a circle and non-discrete control group.
   - The groups $\mathcal{K}_3$ have limit set the whole $\mathbb{P}^2_{\mathbb{C}}$.
(2) The groups $\mathcal{K}_4$, $\mathcal{K}_5$ and $\mathcal{K}_6$ are obtained by a different type of extensions. These also have limit set the whole of $\mathbb{P}^2_{\mathbb{C}}$ and the three classes are distinguished by the rank of their control groups, which are always non-discrete.

2.5.1. Abelian Kodaira groups

These are Abelian subgroups of fundamental groups of Kodaira surfaces; they are finite extensions of dual torus groups of type I.

Definition 2.12. A Kodaira group is a discrete group in $\mathrm{PSL}(3, \mathbb{C})$ such that each element in the group can be written in the form:

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}.$$

We have:

Lemma 2.13. *Let G be a Kodaira group, then G is, weakly-controllable and isomorphic to $Ker(G) \oplus C(G)$ where $C(G)$ is the control group and $Ker(G)$ is the kernel of the control morphism. Also:*

- *We have $Rank\, G \leq 4$*
- *The Kulkarni limit set is a line.*
- *Its complement Ω_{Kul} coincides with the equicontinuity set and is the largest open set on which the group acts properly discontinuously.*
- *The Conze-Guivarc'h limit set of the action on the dual $\check{\mathbb{P}}^2_{\mathbb{C}}$ is a point, the dual of Λ_{Kul}, and it is the unique minimal set.*

Proof. That the group is a direct sum as stated is immediate. The claim about the rank follows from [30]. Now let $(g_n) \subset G$ be a sequence of distinct elements, then there exist sequences $(a_n), (b_n) \subset \mathbb{C}$ such that:

$$g_n = \begin{bmatrix} 1 & a_n & b_n \\ 0 & 1 & a_n \\ 0 & 0 & 1 \end{bmatrix}.$$

This implies that there exist $a, b, c, d \in \mathbb{C}$ satisfying: $|a| + |b| \neq 0$, $|c| + |d| \neq 0$ and

$$g_n \xrightarrow[n \to \infty]{} g = \begin{bmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} ; \text{ and } g_n^* \xrightarrow[n \to \infty]{} h = \begin{bmatrix} 0 & 0 & 0 \\ c & 0 & 0 \\ d & c & 0 \end{bmatrix}.$$

Thus $Ker(g) \subset \overleftrightarrow{e_1, e_2}$. The rest of the proof is as in the elliptic case. □

The next result enables us to provide a normal form for the Kodaira groups.

Lemma 2.14. *A group G is an Abelian Kodaira group if and only if there is $W \subset \mathbb{C}$, an additive discrete subgroup, $R \subset \mathbb{C}$ is an additive subgroup and $L : R \to \mathbb{C}$ a group morphism such that $Rank(W) + Rank(R) \leq 4$,*

$$G = \mathcal{K}_0(W, R, L) = \left\{ \begin{bmatrix} 1 & a & L(a) + a^2/2 + w \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} : a \in R, w \in W \right\},$$

and

$$\lim_{n \to \infty} L(x_n) + w_n = \infty$$

for every sequence $(w_n) \in W$ and every sequence $(x_n) \subset R$ converging to 0.

As an example, consider $w_1 = 1, w_2 = \sqrt{2}, w_3 = e^{\pi i/4}, w_4 = \sqrt{2}e^{\pi i/4}$, and let W be $Span_{\mathbb{Z}}\{w_1, w_2, w_3, w_4\}$. Define $L : W \to \mathbb{C}$ by setting $L(1) = 2^{-1}$, $L(\sqrt{2}) = \sqrt{2} - 1$, $L(e^{\pi i/4}) = i + 2^{-1}$, $L(\sqrt{2}e^{\pi i/4}) = \sqrt{2}i + 1$, and then extend by linearity:

$$L\left(\sum_{j=1}^{4} n_j w_j\right) = \sum_{j=1}^{4} n_j L(w_j).$$

Then

$$\mathcal{K}_0(W, L) = \left\{ \begin{bmatrix} 1 & a & L(a) + a^2 2^{-1} \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} : a \in W \right\}$$

is weakly-controllable and isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. If we let Π be its control morphism, then a straightforward computation shows that its kernel $Ker(\Pi|_{\mathcal{K}_0(W,L)})$ is trivial and the control group is a dense subgroup of $\mathbb{C}$.

Similarly, let W be now $Span_{\mathbb{Z}}(\{1, \sqrt{2}, e^{\pi i/4}\})$ and define $L : W \to \mathbb{C}$ as in the previous example. Then,

$$\left\{ \begin{bmatrix} 1 & x & L(x) + x^2/2 + ik \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} : x \in W, k \in \mathbb{Z} \right\}$$

is a weakly-controllable discrete group isomorphic to $\mathbb{Z}^3 \oplus \mathbb{Z}$. We find that the kernel of the projection to the control group is isomorphic to $\mathbb{Z}$ and the control group is non-discrete and isomorphic to $\mathbb{Z}^2$.

2.5.2. Extended Kodaira groups $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$

Let W be a non-trivial additive subgroup of $\mathbb{C}$ and define a normal form:

$$h_w = \begin{bmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \; w \in W - \{\mathbf{0}\} .$$

Definition 2.15. An *extended Kodaira group* $\mathcal{K}_i$, $i = 1, 2, 3$, is a discrete group generated by a normal form h_w and the normal forms γ_1 and $\mathfrak{g}$

$$\mathfrak{g} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} ; \quad \gamma_1 = \begin{bmatrix} 1 & x + z & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} ; \; x, y, z \in \mathbb{C} .$$

That is: $\mathcal{K}_i = \langle h_w, \mathfrak{g}, \gamma_1 \rangle$.

Lemma 2.16. *We have:*

(1) *The group is $\mathcal{K}_1 \Leftrightarrow$ its control group is discrete $\Leftrightarrow z \in \mathbb{C} \setminus \mathbb{R}$;*
(2) *The group is $\mathcal{K}_2 \Leftrightarrow$ its control group is non-discrete and W has rank 1 $\Leftrightarrow W$ has rank 1 and $z \in \mathbb{R} \setminus \mathbb{Q}$;*
(3) *The group is $\mathcal{K}_3 \Leftrightarrow W$ has rank > 1 and $z \in \mathbb{R} \setminus \mathbb{Q}$. In this case the control group is automatically non-discrete.*

As an example of type III groups take $W = \{(m + ni, k + li) \in \mathbb{C}^2 : k, l, m, n \in \mathbb{Z}\}$, $a = b = 0$ and $c = i$, we get $\text{Heis}(3, \mathbb{Z}[i])$, the Heisenberg group with coefficients in $\mathbb{Z}[i]$.

Lemma 2.17. *The non- Kodaira groups $\mathcal{K}_i$, $i = 1, 2, 3$, can be written as:*

$$\mathcal{K}_i = \left\{ \begin{bmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^n \begin{bmatrix} 1 & x + z & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}^m : m, n \in \mathbb{Z}, w \in W \right\}$$

Hence these are semi direct products of the form $(\mathbb{Z}^{Rank(W)} \rtimes \mathbb{Z}) \rtimes \mathbb{Z}$. The kernel of the control morphism is isomorphic to $\mathbb{Z}^{Rank(W)}$. The control group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and it is discrete if and only if the group is of type $\mathcal{K}_1$.

Proof. The proof is a direct consequence of Theorem 5.15. □

Lemma 2.18. *For the non- Kodaira groups $\mathcal{K}_1$ and $\mathcal{K}_2$ one has:*

- *The Kulkarni discontinuity region coincides with the equicontinuity set and is the largest open set on which the group acts properly discontinuously.*
- *The Conze-Guivarc'h limit set is the dual of Λ_{Kul}. It is a point if the group is $\mathcal{K}_1$ and a real projective line if the group is $\mathcal{K}_2$.*

Proof. The proof is similar to the other cases. Let $(g_n) \subset \mathcal{K}_1$ be a sequence of distinct elements. Then there exist sequences $(k_n), (m_n) \subset \mathbb{Z}$ and $(w_n) \subset W$ such that: g_n is:

$$\begin{bmatrix} 1 & k_n+m_n(x+z) & \frac{1}{2}\big((k_n+m_nz)(k_n+m_nz-1)+m_n(2y+z(-x+1-z))+2w_n+m_n^2zx\big) \\ 0 & 1 & k_n+m_nz \\ 0 & 0 & 1 \end{bmatrix},$$

and the inverse transpose matrix g_n^* is:

$$\begin{bmatrix} 1 & 0 & 0 \\ -k_n-m_nx-m_nz & 1 & 0 \\ 1/2\big(k_n+k_n^2-2w_n-2m_ny+2m_nk_n(x+z)+m_n(1+m_n)z(x+z)\big) & -k_n-m_nz & 1 \end{bmatrix}.$$

Case 1. $(k_n), (m_n)$ are eventually constant. In this case $w_n \xrightarrow[n \to \infty]{} \infty$ and therefore

$$g_n \xrightarrow[n \to \infty]{} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In the sequel we will assume that either (k_n) or (m_n) is a sequence of distinct elements.

Case 2. $k_n + m_n z \xrightarrow[n \to \infty]{} u \in \mathbb{C}$. So $z \in \mathbb{R} - \mathbb{Q}$ and therefore W has rank 1 and $k_n, m_n \xrightarrow[n \to \infty]{} \infty$. Let $w \in \mathbb{C}^*$ be the generator of W and define $\rho_n = \max\{|m_n|, 2^{-1}|m_n(2y + z(-x + 1 - z)) + 2w_n + m_n^2 zx|\}$, so we can assume that there are $a_1, b_1 \in \mathbb{C}$ such that

$$\rho_n^{-1}(m_n, 2^{-1}(m_n(2y + z(-x + 1 - z)) + 2w_n + m_n^2 zx)) \xrightarrow[n \to \infty]{} (a_1, b_1).$$

Thus

$$g_n \xrightarrow[n \to \infty]{} \begin{bmatrix} 0 & a_1x & b_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = g.$$

For simplicity assume $a_1 \neq 0$, so there is $l \in \mathbb{Z}$ and $(l_n) \subset \mathbb{Z}$ such that

$$2b_1 a_1^{-1} = 2y + z(-lw + 1 - z) + w \lim_{n \to \infty} (2l_n m_n^{-1} + m_n zl)$$

where $r = \lim_{n \to \infty} (2l_n m_n^{-1} + m_n zl) \in \mathbb{R}$. Now a straightforward computation shows:

$$Ker(g) = \overleftarrow{e_1, (2lw)^{-1}(2y + z(1 - z) + w(r - lz))e_2 - e_3} .$$

Case 3. $h_n = k_n + m_n z \xrightarrow[n \to \infty]{} \infty$. Let us define:

$$\rho_n = \max \left\{ |h_n + m_n x|, |\tfrac{1}{2}(h_n(h_n - 1) + m_n(2y + z(1 - x - z)) + 2w_n + m_n^2 zx)|, |h_n| \right\}$$

Then we can assume that there are $a_1, b_1, c_1 \in \mathbb{C}$ such that $|a_1| + |b_1| + |c_1| \neq 0$ and

$$\begin{aligned}
\rho_n^{-1}(k_n + m_n(x + z)) &\xrightarrow[n \to \infty]{} a_1 \\
\rho_n^{-1}(\tfrac{1}{2}(h_n(h_n - 1) + m_n(2y + z(1 - x - z)) + 2w_n + m_n^2 zx)) &\xrightarrow[n \to \infty]{} b_1 \qquad (2.2) \\
\rho_n^{-1}(k_n + m_n z) &\xrightarrow[n \to \infty]{} c_1 .
\end{aligned}$$

Hence

$$g_n \xrightarrow[n \to \infty]{} \begin{bmatrix} 0 & a_1 & b_1 \\ 0 & 0 & c_1 \\ 0 & 0 & 0 \end{bmatrix} = g .$$

If $c_1 \neq 0$ we get that $Ker(g)$ is either a point or $\overleftrightarrow{e_1, e_2}$, so let us assume $c_1 = 0$. Under this assumption by equation (2.2) we deduce:

$$\begin{aligned}
\rho_n^{-1} m_n &\xrightarrow[n \to \infty]{} a_1 x^{-1} , \\
\rho_n^{-1}(\tfrac{1}{2}((k_n + m_n z)^2 + 2w_n + m_n^2 zx)) &\xrightarrow[n \to \infty]{} b_1 - a_1 2^{-1} x^{-1}(2y + z(-x + 1 - z)).
\end{aligned}$$

At this point observe that in this case $a_1 = 0$ implies $Ker(g) = \overleftrightarrow{e_1, e_2}$, so we will assume $a_1 \neq 0$.

Claim 1. We have $z \in \mathbb{R}$. Observe that

$$\lim_{n \to \infty} m_n^{-1}(k_n + m_n z) = \lim_{n \to \infty} (\rho_n)^{-1}(k_n + m_n z) \cdot \lim_{n \to \infty} \frac{\rho_n}{m_n} = a_1^{-1} \cdot 0 = 0 .$$

Thus $\lim_{n \to \infty} m_n^{-1} k_n = -z \in \mathbb{R}$.

As a consequence of the previous claim and Lemma 2.17 we deduce that W has rank 1 and $z \in \mathbb{R} - \mathbb{Q}$. Moreover, by our previous analysis the only interesting case is $W \not\subset \mathbb{R}$. As before let w be the generator of W, thus there are $(l_n)_{n \geq 0} \subset \mathbb{Z}$ such that

$$g = \begin{bmatrix} 0 & x & 2^{-1}(2y + z(-l_0w + 1 - z) + s) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$

where $s = \lim_{n\to\infty} m_n^{-1}(\frac{1}{2}((k_n + m_n z)^2 + 2l_n w + m_n^2 z l_0 w))$. Now observe that the following limits exist and they are finite.

$$\lim_{n\to\infty} \frac{1}{m_n}(k_n + m_n z)^2; \quad 2s_2 = \lim_{n\to\infty} m_n^{-1}(2l_n + m_n^2 z l_0) .$$

Thus $s = s_1 + s_2 w$ and

$$g = \begin{bmatrix} 0 & l_0 w & 2^{-1}(2y + z(1 - z) + s_1 + (s_2 - l_0)w) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$

which concludes the proof. □

2.5.3. Extended Kodaira groups $\mathcal{K}_4$, $\mathcal{K}_5$

These are similar to the previous groups, but with one and two more generators, respectively. We introduce the following normal forms:

$$\gamma_2 = \begin{bmatrix} 1 & a+c & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} , \quad \gamma_3 = \begin{bmatrix} 1 & d+f & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} ,$$

with $a, b, c, d, e, f \in \mathbb{C}$. Notice that $\gamma_1, \gamma_2, \gamma_3$ are matrices of the same type evaluated on different parameters. In each case we will specify the conditions on all these parameters.

Let $k, l, m \in \mathbb{Z}$, $w \in W$, with W as above, and $x, a, b, c, d, e, f \in \mathbb{C}$ satisfy: $a \neq 0$, $\{a, d, af - dc\} \subset W$, and let $\mathcal{K}_4$ be the group depending on all these parameters, defined by

$$\mathcal{K}_4 = \left\{ h_w \cdot g^k \cdot \gamma_1^m \cdot \gamma_2^n \ : \ k, m, n \in \mathbb{Z}, w \in W \right\} .$$

We assume further that for every real line $\ell \subset \mathbb{C}$ passing through the origin we have $rank(\ell \cap Span_{\mathbb{Z}}\{1, c, f\}) \leq 2$. This condition springs from [32] where the author considers the density properties of finitely generated subgroups of rational points on a commutative algebraic group over a number field. Additionally the following restrictions should be imposed over the coefficients in order to get discrete groups:

(1) if $d = 0$ then $f \notin \mathbb{R}$;
(2) if $ad^{-1} \notin \mathbb{R}$ then there are $r_1, r_2 \in \mathbb{Q}$ such that

$$c = \frac{a(f - r_1)}{d} - r_2 .$$

We now take the previous $\mathcal{K}_4$-groups and add one more generator. Define

$$\mathcal{K}_5 = \{h_w \cdot g^k \cdot \gamma_1^m \cdot \gamma_2^m \cdot \gamma_3^l \ : \ k, m, n, l \in \mathbb{Z}, w \in W\},$$

where $x, a, b, c, d, e, f, g, h, j \in \mathbb{C}$ are subject to the conditions: $a(|d| + |g|) \neq 0$, $\{a, d, g, dj - gf, af - cd, aj - cg\} \subset W$. Furthermore:

(1) If $g = 0$, then there are $r_0, r_1, r_2, r_3 \in \mathbb{Q}$ such that $r_1 \neq 0$ and

$$(r_2 - r_0)^2 + 4r_1 r_3 < 0;$$

such that:

$$j = \frac{r_2 + r_0 \pm \sqrt{(r_2 - r_0)^2 + 4r_1 r_3}}{2};$$

$$a = d \frac{r_2 - r_0 \pm \sqrt{(r_2 - r_0)^2 + 4r_1 r_3}}{2r_1};$$

$$c = (f - r_4) \frac{r_2 - r_0 \pm \sqrt{(r_2 - r_0)^2 + 4r_1 r_3}}{2r_1} - r_5$$

(2) If $ad^{-1} \notin \mathbb{R}$, then there are $r_1, r_2, s_1, t_1, s_2, t_2, s_3, t_3 \in \mathbb{R}$ such that:

$$g = r_1 a + r_2 d, \quad r_2 t_2 \neq t_3,$$

$$f = \frac{A_2 \pm (c + t_2) \sqrt{A_1}}{2(r_2 t_2 - t_3)}, \quad j = \frac{A_3 \pm (cr_2 + t_3) \sqrt{A_1}}{2(r_2 t_2 - t_3)},$$

where:

$$A_1 = (-r_2 s_2 + r_1 t_2 - s_3 - t_1)^2 - 4(r_2 s_1 t_2 - r_1 s_2 t_3 + r_2 s_2 s_3 - r_1 t_1 t_2 + s_3 t_1 - s_1 t_3),$$

$$A_2 = -cr_2 s_2 - cr_1 t_2 + cs_3 - ct_1 + r_2 s_2 t_2 - r_1 t_2^2 + s_3 t_2 - 2s_2 t_3 - t_1 t_2,$$

$$A_3 = r_2 (cr_1 t_2 + s_3 (c + 2t_2) - ct_1 - s_2 t_3) + t_3 (-r_1 (2c + t_2) - s_3 - t_1) - cr_2^2 s_2.$$

These groups have the following properties:

Lemma 2.19. *The $\mathcal{K}_4$ and $\mathcal{K}_5$ groups are all weakly-controllable, discrete, with kernel isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and their control group is non-discrete and isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ respectively. Moreover the Kulkarni limit set is $\mathbb{P}^2_{\mathbb{C}}$ and the equicontinuity regions is empty.*

Proof. For the proof of this lemma see Lemma 5.24 and Propositions 5.25, 5.26. $\square$

Remark 2.20. We notice that in these families we can have examples where the control group $\Pi(\mathcal{K}_i)$ is non-discrete but is not dense in $\mathbb{C}$, as well as examples where $\Pi(\mathcal{K}_i)$ is

dense in $\mathbb{C}$. The control group by definition is a subgroup of $\mathrm{PSL}(2,\mathbb{C})$, yet, in the cases we consider here, each element in the control group is a translation, so we can think of the control group as being an additive subgroup of $\mathbb{C}$.

For instance, taking $W = \mathbb{Z}[i]$, $x = b = e = d = 0$, $f = i$ and c an irrational number, we generate a discrete group with a non-discrete dense subgroup of $\mathbb{C}$ as control group. However, taking $W = \mathbb{Z}[i]$, $x = a = b = e = 0$, $c = i$, $d = 1$ and $f = r + is$, where $r, s \in \mathbb{R}$ satisfy that $\{1, r, s\}$ is a $\mathbb{Q}$-linearly independent set, then the corresponding discrete group has a dense subgroup of $\mathbb{C}$ as control group. This shows that unlike the 1-dimensional case where purely parabolic groups have very simple dynamics, in dimension 2 the two different dynamics described above, both fairly rich, exist for control groups of purely parabolic groups. This type of behavior is important when studying non-discrete subgroups of Lie groups, see for instance [27,32].

3. The classification theorems

We now provide a complete classification of the purely parabolic discrete subgroups of $\mathrm{PSL}(3,\mathbb{C})$.

Theorem 3.1. *Let $G \subset \mathrm{PSL}(3,\mathbb{C})$ be a complex Kleinian discrete subgroup. Then G does not contain loxodromic elements if, and only if, there exists a normal subgroup $G_0 \subset G$ of finite index such that G_0 is purely parabolic and it is conjugate to one (and only one) of the following groups:*

(1) *An Elliptic group as in Subsection 2.1.*
(2) *A Torus group, as in Subsection 2.2.*
(3) *A dual Torus group of type I and II, as in Definition 2.4.*
(4) *A Kleinian Inoue group, as in Theorem 2.7.*
(5) *An Kodaira group, as in Example 2.5.1.*
(6) *An extended Kodaira group K_1 or K_2, as in Definition 2.15.*

Theorem 3.2. *Let $G \subset \mathrm{PSL}(3,\mathbb{C})$ be a discrete subgroup which is not Kleinian. Then G does not contain loxodromic elements if, and only if, there exists a normal subgroup $G_0 \subset G$ of finite index such that G_0 is purely parabolic and it is conjugate to one (and only one) of the following groups:*

(1) *A dual Torus group of type III, as in Definition 2.4.*
(2) *A discrete non Kleinian Inoue group, as in Theorem 2.7.*
(3) *An extended Inoue group, as in Definition 2.8.*
(4) *An extended Kodaira group $\mathcal{K}_3$, as in Definition 2.15.*
(5) *An extended Kodaira group $\mathcal{K}_4$ or $\mathcal{K}_5$, as in Subsection 2.5.3.*

The rest of this article is devoted to proving Theorems 3.1 and 3.2.

Concerning the dynamics we have:

Corollary 3.3. *F Let $G \subset \mathrm{PSL}(3, \mathbb{C})$ be a complex Kleinian group. Then:*

(1) *The Kulkarni limit set Λ_{Kul} is:*
   - *A pencil of lines over a circle if it is a dual torus group of type II, an Inoue group or extended Kodaira groups $\mathcal{K}_2$.*
   - *One line otherwise.*
(2) *Concerning the Conze-Guivarc'h set Λ^*_{CoG} of the action on the dual $\check{\mathbb{P}}^2_\mathbb{C}$:*
   - *Is either a point or a real projective line.*
   - *It is a minimal whenever is a single point.*

Proof. The proof of part (1) goes as follows. By Theorem 3.2, G is virtually conjugated to one and only one of the groups given in Theorem 3.2. From Lemma 2.5, Theorem 2.7 and Corollary 2.10 we know the groups with limit set a cone of lines over a circle: these are dual torus group type II, Kleinian Inoue and $\mathcal{K}_2$ groups and for the remaining ones in Theorem 3.2 its limit set is a line.

In order to proof of part (2) we apply Lemmas 2.2, 2.5, 2.13, 2.18, Theorem 2.7 and Corollary 2.10, the Conze-Guivarc'h limit set is either a real projective line or a point.   □

Concerning the region of discontinuity, this is empty in all non-Kleinian cases. In the Kleinian case we have:

Corollary 3.4. *Let $G \subset \mathrm{PSL}(3, \mathbb{C})$ be a purely parabolic complex Kleinian group. Then Ω_{Kul} is biholomorphic to:*

(1) $\mathbb{C}^2$ *if Λ_{Kul} is a line.*
(2) *In other case is $\mathbb{C} \times (\mathbb{H}^+ \cup \mathbb{H}^-)$ where $\mathbb{H}^+$ is the upper half plane in $\mathbb{C}$ and $\mathbb{H}^-$ is the lower half plane in $\mathbb{C}$.*

Proof. The proof follows immediately from Theorem 3.3 item (1) because when the limit set is a line, then its complement in $\mathbb{P}^2_\mathbb{C}$ is biholomorphic to $\mathbb{C}^2$, and if the limit set is a pencil of lines over a circle, then its complement is biholomorphic to $\mathbb{C} \times (\mathbb{H}^+ \cup \mathbb{H}^-)$.   □

4. Dynamics of triangularizable groups without loxodromic elements

In this section we use techniques of dynamical systems in order to show that discrete subgroups of $\mathrm{PSL}(3, \mathbb{C})$ without loxodromic elements are triangularizable, see Theorem 4.3. Moreover we show that these groups are either subgroups of $\mathrm{Heis}(3, \mathbb{C})$ or they can be described in a very precise way, see Theorem 4.5.

4.1. Purely parabolic groups are virtually triangularizable

Recall that a discrete group $G \subset \mathrm{PSL}(3,\mathbb{C})$ acts strongly irreducibly on $\mathbb{P}^2_{\mathbb{C}}$ if there are no points or lines with finite orbit. Also a group $G \subset \mathrm{PSL}(3,\mathbb{C})$ is *affine* if G has an invariant complex line in $\mathbb{P}^2_{\mathbb{C}}$. Now we have:

Lemma 4.1. *If $G \subset \mathrm{PSL}(3,\mathbb{C})$ is a discrete group without loxodromic elements, then G is either affine or weakly-controllable.*

Proof. By [3, Proposition 4.10], discrete groups in $\mathrm{PSL}(3,\mathbb{C})$ acting strongly irreducibly on $\mathbb{P}^2_{\mathbb{C}}$ contain loxodromic elements, so we can assume that there is a non-empty proper subspace $l \subset \mathbb{P}^2_{\mathbb{C}}$ such that l has a finite orbit under the action of G. Observe that by duality we can assume l is a point; let $l_1, \ldots, l_k$ be the orbit of l under G.

Let U be the projective space generated by $\{l_1, \ldots, l_k\}$; clearly U is G-invariant. We claim that U is either a point or a line. Assume, on the contrary, that $U = \mathbb{P}^2_{\mathbb{C}}$. Let $g \in G$ be a parabolic element, then there exist $s \in \{1, \ldots, k\}$ such that $l_s \notin \Lambda_{Kul}(\langle g \rangle)$ then l_s has infinite orbit under the cyclic group $\langle g \rangle$, which is a contradiction. $\square$

If $G \subset \mathrm{PSL}(3,\mathbb{C})$ does not contain loxodromic elements, then Lemma 4.1 implies that G has either an invariant line or an invariant pencil of lines. The following lemma gives restrictions upon the action of G on the invariant line or pencil.

Lemma 4.2. *Let $G \subset \mathrm{PSL}(3,\mathbb{C})$ be a discrete group without loxodromic elements.*

(1) *If G is affine, then the action of G on the invariant line does not contain a subgroup conjugate to a dense subgroup of $\mathrm{SO}(3)$.*
(2) *If G is weakly-controllable, then the control group of G does not contain a subgroup conjugate to a dense subgroup of $\mathrm{SO}(3)$.*

Proof. Let us prove only the case where G is weakly-controllable, the proof in the affine case is similar. As before, let Π be the projection to the control group. Let us proceed by contradiction, let $(g_n)_{n \in \mathbb{N}}$ be an enumeration of G and define $H_m = \Pi(\langle g_1, \ldots g_m \rangle)$. If each group is finite, then by the classification of subgroups in $\mathrm{PSL}(2,\mathbb{C})$ with finite order, we conclude that for m large H_m is either cyclic or dihedral, and therefore the control group $\Pi(G)$ is a subgroup of the infinite dihedral group; this is not possible since $\Pi(G)$ is dense in $SO(3)$. Now, applying Selberg's Lemma to the H_m's, we deduce that $\Pi(G)$ contains an element with infinite order. On the other hand, since $\Pi(G)$ is dense in $SO(3)$ and by Tits alternative we conclude that for m large, H_m contains a rank two free subgroup. To conclude, let $g_1, g_2 \in G$ be such that $\Pi(g_1), \Pi(g_2)$ generate a rank two free group, then $g_1 g_2 g_1^{-1} g_2^{-1}$ has a lift $\rho \in \mathrm{SL}(3,\mathbb{C})$, given by

$$\rho = \begin{pmatrix} 1 & b \\ \mathbf{0} & B \end{pmatrix}$$

where $B \in \mathrm{SO}(3)$ has infinite order and $b \in \mathbb{C}^2$. Clearly ρ is non-diagonalizable, with unitary eigenvalues and infinite order. Therefore G is non-discrete. $\square$

Now we prove the following extension of the Lie-Kolchin Theorem [29], in which we allow the existence of non-unipotent elements.

Theorem 4.3. *Let $G \subset \mathrm{PSL}(3,\mathbb{C})$ be a discrete group without loxodromic elements. Then there exists a normal subgroup G_0 of G with finite index such that G_0 leaves invariant a full flag in $\mathbb{P}^2_\mathbb{C}$. Hence the group G_0 is simultaneously triangularizable.*

Proof. Let $G \subset \mathrm{PSL}(3,\mathbb{C})$ be a discrete group without loxodromic elements. Then by Lemma 4.1 we know that G has a proper non-empty projective subspace p invariant under G. Here we prove the case where p is a point; the other case is analogous, considering a line ℓ as a point in $\check{\mathbb{P}}^2_\mathbb{C}$. Since G does not contain loxodromic elements, neither $\Pi(G)$ does, and by Corollary 13.7 in [17] we have that $\mathbb{P}^1_\mathbb{C} - Eq(\Pi(G))$ is either empty or contains a single point. If $\mathbb{P}^1_\mathbb{C} - Eq(\Pi(G))$ is a single point, then G is simultaneously triangularizable. So we assume $\mathbb{P}^1_\mathbb{C} = Eq(\Pi(G))$. Then Lemma 4.2 implies that $\Pi(G)$ is either finite or it is a subgroup of the infinite dihedral group Dih_∞. If $\Pi(G)$ is finite, it is enough to consider $G_0 = Ker(\Pi|_G)$. If $\Pi(G) \subset Dih_\infty$ then consider $G_0 = \{g \in G : \Pi(g) \in Rot_\infty\}$. $\square$

Corollary 4.4. *Let $G \subset \mathrm{PSL}(3,\mathbb{C})$ be a discrete group without loxodromic elements, then G is virtually finitely generated.*

Proof. Since G does not contain loxodromic elements, we know that G contains a finite index subgroup which is triangularizable and therefore solvable. It is well known that discrete solvable groups are finitely generated, see [2]. $\square$

4.2. A Lie-Kolchin theorem for purely parabolic groups

The following is a slight extension of the Lie-Kolchin Theorem.

Theorem 4.5. *Let G be a purely parabolic discrete group in $\mathrm{PSL}(3,\mathbb{C})$. Then G is either virtually unipotent or it contains a subgroup of finite index which is conjugate to:*

$$G = \left\{ \begin{bmatrix} 1 & w & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \eta(w) \end{bmatrix} : w \in W,\ n \in \mathbb{Z} \right\},$$

with W a discrete additive subgroup of $\mathbb{C}$ and $\eta : W \to \mathbb{S}^1$ a group morphism.

This subsection is divided into four parts: in 4.2.1 and 4.2.2 we show that every discrete solvable group with an irrational ellipto-parabolic element is commutative, see Corollary 4.14. In Subsection 4.2.3 we give a list of all Commutative Lie groups of $\mathrm{PSL}(3,\mathbb{C})$. Finally, in 4.2.4 we prove Theorem 4.5 and we also prove Theorem 5.20.

4.2.1. Solvable groups with an irrational ellipto-parabolic element

Lemma 4.6. *Let $\alpha \in \mathbb{S}^1$ be an element with infinite order and $a, b, c, x, y, z \in \mathbb{C}$. If x, y are not both zero, then the group*

$$G = \left\langle g_1 = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, g_2 = \begin{bmatrix} 1 & a & b \\ 0 & \alpha^3 & c \\ 0 & 0 & 1 \end{bmatrix} \right\rangle,$$

is non-discrete.

Proof. Let $h \in \mathrm{PSL}(3, \mathbb{C})$ be given by:

$$h = \begin{bmatrix} 1 & a(1 - \alpha^3)^{-1} & b \\ 0 & 1 & -c(1 - \alpha^3)^{-1} \\ 0 & 0 & 1 \end{bmatrix}.$$

A straightforward computation shows:

$$hg_2h^{-1} = \begin{bmatrix} 1 & 0 & b + ac(1 - \alpha^3)^{-1} \\ 0 & \alpha^3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad hg_1h^{-1} = \begin{bmatrix} 1 & x & z + cx(1 - \alpha^3)^{-1} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

We take $a = c = 0$. Set $g_n = g_2^n g_1 g_2^{-n}$ and observe that we have:

$$g_n = g_2^n g_1 g_2^{-n} = \begin{bmatrix} 1 & x\alpha^{-3n} & z \\ 0 & 1 & y\alpha^{3n} \\ 0 & 0 & 1 \end{bmatrix}.$$

Clearly (g_n) contains a convergent sequence of distinct elements, proving the lemma. □

Lemma 4.7. *Let $\alpha \in \mathbb{S}^1$ be an element with infinite order and $x, y, z, \beta, \mu, \nu \in \mathbb{C}$. If x, y are not both zero, then the group*

$$G = \left\langle g_1 = \begin{bmatrix} 1 & z & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, g_2 = \begin{bmatrix} 1 & \beta & \mu \\ 0 & 1 & \nu \\ 0 & 0 & \alpha^{-3} \end{bmatrix} \right\rangle,$$

is non-discrete.

Proof. Notice first that $\beta = 0$ implies that g_2 is an elliptic element with infinite order, which makes G non-discrete. So we assume that $\beta \neq 0$ and G is discrete. An easy computation shows:

$$G_0 = [G, G] = \left\{ \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} : (a, b) \in \mathfrak{L} \right\},$$

where $\mathfrak{L}$ is a discrete additive subgroup of $\mathbb{C}^2$. Consider $g = [g_{ij}] = [g_2, [g_2, g_1]]$. Then $g \in G_0$ and $g_{13}g_{23} \neq 0$, so after conjugating with an upper triangular element, if necessary, we can assume that $(1,1) \in \mathfrak{L}$. A straightforward computation shows:

$$
g_2^n \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} g_2^{-n} = \begin{bmatrix} 1 & 0 & \alpha^{3n}(n\beta+1) \\ 0 & 1 & \alpha^{3n} \\ 0 & 0 & 1 \end{bmatrix};
$$

thus $\{\alpha^{3n}(n\beta+1, 1) : n \in \mathbb{Z}\} \subset \mathfrak{L}$. Now we claim:

Claim 1. The set $A = \{\alpha^{3n}(n\beta+1, 1) : n = 0, 1, 2, 3\}$ is $\mathbb{R}$-linearly independent. Assume, on the contrary, that there are $r_0, r_1, r_2, r_3 \in \mathbb{R}$ not all equal to 0, such that:

$$
\begin{aligned}
0 &= \sum_{j=0}^{3} r_j \alpha^{3j}(j\beta+1, 1) \\
&= \left(\sum_{j=0}^{3} r_j \alpha^{3j}, 0 \right) + \sum_{j=0}^{3} \beta r_j \alpha^{3j}(j, 1) \\
&= \sum_{j=0}^{3} \beta r_j \alpha^{3j}(j, 1).
\end{aligned}
$$

Thus $\{\alpha^{3j}(j, 1) : j = 0, 1, 2, 3\}$ is $\mathbb{R}$-linearly dependent, which is a contradiction.

Claim 2. $A = \{\alpha^{3j}(j, 1) : j = 0, 1, 2, 3, 4\}$ is a $\mathbb{Q}$-linearly dependent set. Observe that $B = \{\alpha^{3j}(\beta j+1, 1) : j = 0, 1, 2, 3, 4\}$ is $\mathbb{Q}$-linearly dependent; then using similar arguments as in the previous claim, we get that A is $\mathbb{Q}$-linearly dependent.

Claim 3. There is $d \in \mathbb{C}^*$ such that $(d, 0) \in \mathfrak{L}$. By Lemma 7.2, there exists $c \in \mathbb{C}^*$ and $m_0, \ldots, m_5 \in \mathbb{Z}$ such that

$$
(c, 0) = \sum_{j=0}^{5} m_j \alpha^{3j}(j, 1),
$$

thus

$$
(c\beta, 0) = \sum_{j=0}^{5} m_j \alpha^{3j}(j\beta+1, 1).
$$

Finally, let $(m_n) \subset \mathbb{Z}$ be such that (α^{3m_n}) is a sequence of distinct elements which converge to 1 and $d \in \mathbb{C}^*$ is such that $(d, 0) \in \mathfrak{L}$. Then

$$
g_m = g_1^{m_n} \begin{bmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} g_1^{-m_n} = \begin{bmatrix} 1 & 0 & \alpha^{3m_n}d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[n\to\infty]{} \begin{bmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

which is a contradiction, so G is non-discrete. $\square$

Lemma 4.8. *Let $\alpha \in \mathbb{S}^1$ be an element with infinite order and $x, y, z, \beta, \mu, \nu \in \mathbb{C}$. If x, y are not both zero, then the group*

$$G = \left\langle g_1 = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}, \ g_2 = \begin{bmatrix} \alpha^{-3} & \beta & \mu \\ 0 & 1 & \nu \\ 0 & 0 & 1 \end{bmatrix} \right\rangle$$

is non-discrete.

Proof. Consider the group morphism $\rho : \mathrm{PSL}(3,\mathbb{C}) \to \mathrm{PSL}(3,\mathbb{C})$ given by $\rho([M]) = (M^t)^{-1}$, here M^t denotes the transpose matrix of M. We claim that $\rho(G)$ is non-discrete, which will prove the lemma. For this, notice that Lemma 4.7 implies:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}^t \right)^{-1} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -z & zx - y \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} \alpha^{-3} & \beta & \mu \\ 0 & 1 & \nu \\ 0 & 0 & 1 \end{bmatrix}^t \right)^{-1} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -\nu & \alpha^3(\beta\nu - \mu) \\ 0 & 1 & -\beta\alpha^3 \\ 0 & 0 & \alpha^3 \end{bmatrix},$$

and the result follows. □

4.2.2. *Commutator group of solvable discrete groups containing irrational ellipto-parabolic elements*

In the following, if $g \in \mathrm{GL}\,(3,\mathbb{C})$, then g_{ij} will denote the ij-th element of the matrix g.

Definition 4.9. Define a group U_+ in $\mathrm{PSL}(3,\mathbb{C})$ by:

$$U_+ = \left\{ \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ 0 & g_{22} & g_{23} \\ 0 & 0 & g_{33} \end{bmatrix} : g_{11}g_{22}g_{33} = 1 \right\}$$

and the group morphisms $\Pi^* : U_+ \to Mob(\mathbb{C})$ and $\lambda_{12}, \lambda_{23}, \lambda_{13} : U_+ \to \mathbb{C}^*$, given by:

$$\Pi^*([g_{ij}])z = g_{11}g_{22}^{-1}z + g_{12}g_{22}^{-1} \, ,$$
$$\lambda_{12}([g_{ij}]) = g_{11}g_{22}^{-1} \, ,$$
$$\lambda_{23}([g_{ij}]) = g_{22}g_{33}^{-1} \, ,$$
$$\lambda_{13}([g_{ij}]) = g_{11}g_{33}^{-1} \, .$$

Notice that the elements in U_+ are equivalence classes of matrices. Yet, since different representatives of the same projective transformation differ by multiplication by a scalar, the above homomorphisms are all well-defined.

Lemma 4.10. *Let $G \subset U_+$ be a discrete group, then G contains a finite index torsion free subgroup G_0 such that the following groups are torsion free: the control group $\Pi(G_0)$, the dual control group $\Pi^*(G_0)$, $\lambda_{12}(G_0)$, $\lambda_{13}(G_0)$ and $\lambda_{23}(G_0)$.*

Proof. By Selberg's lemma, we can assume that G is torsion free. Now consider the control group $\Pi|_G$. Notice that:

Step 1. We can apply Selberg's lemma to the group $\Pi(G)$ to get a finite index subgroup $G_1 \subset \Pi(G)$ which is torsion free.

Step 2. Define $\widetilde{G} = f^{-1}(G_1)$ and notice this is a finite index subgroup in G.

Step 3. Using Selberg's lemma again, we get a torsion free, finite index subgroup G_2 of $\widetilde{G}$.

Step 4. Notice that G_2, which is torsion free, has finite index in G and its control group $\Pi(G_2)$ also is torsion free. This proves the first statement.

We may now follow this same process with the groups G_2 and $\Pi^*|_{G_2}$, granting the existence of a finite index, torsion free subgroup G_3 of G for which $\Pi^*(G_3)$ is torsion free. Notice this same process can be applied to the morphisms $\lambda_{12}(G_0)$, $\lambda_{13}(G_0)$ y $\lambda_{23}(G_0)$, thus proving the lemma. $\quad\square$

The following corollary is an immediate consequence of Lemma 4.10 and it is of interest in itself:

Corollary 4.11. *Let G be an upper triangular discrete subgroup of $\mathrm{SL}(3,\mathbb{C})$. Then G has a finite index subgroup that does not contain neither elliptic nor rational screws nor rational ellipto-parabolic elements.*

We refer to [16, Chapter 4] for the definition of rational screws, which are all loxodromic elements.

Lemma 4.12. *Let $G \subset U_+$ be a discrete group such that the groups $\lambda_{12}(G), \lambda_{23}(G), \lambda_{13}(G)$ are torsion free. If $g \in G$ is an irrational ellipto-parabolic element, then g belongs to the center of G, i.e., g commutes with every element of G.*

Proof. Assume on the contrary, that there exists an element $h = [h_{ij}] \in G$ such that $[g, h] \neq Id$. Then there are $x, y, z \in \mathbb{C}$ such that

$$[g, h] = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}.$$

Now consider the following cases:

Case 1. We have $o(\lambda_{12}(g)) = o(\lambda_{23}(g)) = \infty$, where $o(\ )$ means the order. Since G is discrete we deduce that $\lambda_{13}(g) = 1$. By Lemma 4.6 we have $x = z = 0$ but $y \neq 0$. We deduce $\Pi^*[g, h] = \Pi[g, h] = Id$, $g_{13} \neq 0$ and $o(\lambda_{13}(h)) = \infty$, therefore

$$h^n g h^{-n} = \begin{bmatrix} g_{11} & g_{12} & (\lambda_{13}(h))^n g_{13} \\ 0 & g_{11}^{-2} & g_{23} \\ 0 & 0 & g_{11} \end{bmatrix}.$$

So the sequence $(h^n g h^{-n})_{n \in \mathbb{Z}}$ contains a subsequence of distinct elements which converges to a projective transformation and G is non-discrete.

Case 2. We have $o(\lambda_{12}(g)) = \infty$ and $\lambda_{23}(g) = 1$. Since G is discrete we deduce that $g_{23} \neq 0$. By Lemma 4.8, we deduce $x = y = 0$ but $z \neq 0$. Therefore $\Pi^*[g, h] = Id$ and $o(\lambda_{23}(h)) = \infty$. Then:

$$h^n g h^{-n} = \begin{bmatrix} g_{11}^{-2} & g_{12} & g_{13} \\ 0 & g_{11} & \lambda_{23}^n(h) g_{23} \\ 0 & 0 & g_{11} \end{bmatrix}.$$

Thus $(h^n g h^{-n})_{n \in \mathbb{Z}}$ contains a subsequence of distinct elements which converges to a projective transformation.

Case 3. We have $o(\lambda_{23}(g)) = \infty$ and $\lambda_{12}(g) = 1$. Again, since G is discrete we deduce $g_{12} \neq 0$. By Lemma 4.7 we deduce $x = z = 0$ but $y \neq 0$. Therefore $\Pi[g, h] = Id$ and $o(\lambda_{12}(h)) = \infty$. As in the previous cases we get:

$$h^n g h^{-n} = \begin{bmatrix} g_{11} & \lambda_{23}^n(h) g_{12} & g_{13} \\ 0 & g_{11} & g_{23} \\ 0 & 0 & g_{11}^{-2} \end{bmatrix},$$

so $(h^n g h^{-n})$ contains a subsequence of distinct elements which converges to a projective transformation.

Thus we have shown that under the assumption that G is not commutative we get that G is non-discrete, which is a contradiction. $\square$

Lemma 4.13. *Let $G \subset U_+$ be a discrete group such that the groups $\lambda_{12}(G)$, $\lambda_{23}(G)$, $\lambda_{13}(G)$ are torsion free. If G contains an irrational ellipto-parabolic element, then G is commutative.*

Proof. We consider first the case where $o(\lambda_{12}(g)) = o(\lambda_{23}(g)) = \infty$. Since G is discrete we deduce that $\lambda_{13}(g) = 1$. Then there exists $h \in U_+$ such that

$$hgh^{-1} = \begin{bmatrix} g_{11} & 0 & a \\ 0 & g_{11}^{-2} & 0 \\ 0 & 0 & g_{11} \end{bmatrix},$$

where $a \neq 0$. Since every element $\beta \in G$ commutes with g we have

$$h\beta h^{-1} = \begin{bmatrix} \beta_{11} & 0 & b \\ 0 & \beta_{11}^{-2} & 0 \\ 0 & 0 & \beta_{11} \end{bmatrix}.$$

This shows that G is commutative.

We can apply similar arguments when either $o(\lambda_{12}(g))$ or $o(\lambda_{23}(g))$ is finite to show that in all cases G is commutative. $\quad \square$

The following result is a consequence of Corollary 4.11 and Lemmas 4.12, 4.13

Corollary 4.14. *Every discrete solvable group with an irrational ellipto-parabolic element, is commutative.*

4.2.3. Abelian Lie groups

The following list of Lie groups is used in Theorem 4.16.

Definition 4.15. We set:

$$C_1 = \left\{ \begin{bmatrix} \alpha^{-2} & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{bmatrix} : \alpha \in \mathbb{C}^*, \beta \in \mathbb{C} \right\} , \quad C_2 = \left\{ \begin{bmatrix} 1 & 0 & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{bmatrix} : \beta, \gamma \in \mathbb{C} \right\} ,$$

$$C_3 = \left\{ \begin{bmatrix} 1 & \alpha & \beta \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{bmatrix} : \alpha, \beta \in \mathbb{C} \right\} , \quad C_4 = \left\{ \begin{bmatrix} 1 & \alpha & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : \alpha, \beta \in \mathbb{C} \right\} ,$$

$$C_5 = Diag(3, \mathbb{C}) = \left\{ \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha^{-1}\beta^{-1} \end{bmatrix} : \alpha, \beta \in \mathbb{C}^* \right\} .$$

Theorem 4.16. *Let $G \subset U_+$ be a commutative group. Then G is conjugate to a group $\widetilde{G} \subset C_j$ for some $j = 1, 2, 3, 4, 5$.*

This theorem is proved in the Appendix 6 at the end of this paper.

4.2.4. Proof of Theorem 4.5

Lemma 4.17. *Let $G \subset U_+$ be a discrete torsion free group such that the group $Ker(\Pi|_G)$ is trivial and each element in $g \in G$ has the form*

$$\begin{bmatrix} \alpha^{-2} & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{bmatrix} ,$$

where $| \alpha | = 1$. Then there exists $W \subset \mathbb{C}$ a discrete additive subgroup and a group morphism $\eta : W \to \mathbb{S}^1$ such that:

$$G = \left\{ \begin{bmatrix} \eta(w)^{-2} & 0 & 0 \\ 0 & \eta(w) & \eta(w)w \\ 0 & 0 & \eta(w) \end{bmatrix} : w \in W \right\} .$$

Proof. Let us define $\zeta : G \to \mathbb{C}$ by $\zeta([g_{ij}]) = g_{23}g_{33}^{-1}$. A standard computation shows that ζ is a group morphism and $Ker(\zeta)$ is trivial. Then the following is a well defined group morphism

$$\eta : \zeta(G) \to \mathbb{C}^*$$
$$x \mapsto \pi_{22}(\zeta^{-1}(x)).$$

Clearly

$$G = \left\{ \begin{bmatrix} \eta(w)^{-2} & 0 & 0 \\ 0 & \eta(w) & \eta(w)w \\ 0 & 0 & \eta(w) \end{bmatrix} : w \in \zeta(G) \right\}. \tag{4.1}$$

We claim that the group $\zeta(G)$ is discrete. Assume on the contrary, that $\zeta(G)$ is non-discrete. Then there exists a sequence $(g_n)_{n \in \mathbb{N}} \subset G$ of distinct elements such that $(\zeta(g_n))_{n \in \mathbb{N}}$ is a sequence of distinct elements and $\zeta(g_n) \xrightarrow[n \to \infty]{} 1$. Then

$$g_n = \begin{bmatrix} \eta(\zeta(g_n))^{-2} & 0 & 0 \\ 0 & \eta(\zeta(g_n)) & \eta(\zeta(g_n))\zeta(g_n) \\ 0 & 0 & \eta(\zeta(g_n)) \end{bmatrix} .$$

Since $\eta(G) \subset \mathbb{S}^1$ we deduce that (g_n) contains a convergent subsequence, which is a contradiction. Therefore $\zeta(G)$ is discrete and we can take $W = \zeta(G)$. $\square$

Lemma 4.18. *Let $G \subset \mathrm{PSL}(3, \mathbb{C})$ be a discrete group where each element has the form:*

$$g = \begin{bmatrix} a^{-2} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}.$$

Then G is virtually cyclic.

Proof. Define $\rho_{12} : G \to \mathbb{R}$ by $\rho_{12}(g) = log(|\lambda_{12}|)$, clearly ρ_{12} is a well defined group morphism, $Ker(\rho_{12}) = \{g \in g : |\lambda_{12}| = 1\}$ and $\rho_{12}(G)$ is discrete. Let $G_0 \subset G$ be a torsion free subgroup of G with finite index. Clearly $\rho_{12}|_{G_0}$ is injective and $\rho_{12}(G_0)$ is cyclic. $\square$

Now we complete the proof of Theorem 4.5. Let $G \subset U^+$ be a discrete group which contains an element g_0 which satisfies $\max\{o(\lambda_{12}(g_0)), o(\lambda_{23}(g_0))\} = \infty$. By Lemma 4.10, G contains a finite index subgroup G_0 for which the groups $\lambda_{12}(G_0)$, $\lambda_{23}(G_0)$, $\Pi^*(G_0)$, $\Pi(G_0)$ and G_0 itself, are all torsion free and finitely generated. Then by Lemma 4.12, G_0 is. Therefore by Theorem 4.16 the group G_0 is conjugate to a group $G_0 \subset G$ and G lives either in $Diag(3, \mathbb{C})$ or in C_1. Then the theorem follows from Lemmas 4.17 and 4.18. $\square$

5. Discrete subgroups in Heis $(3, \mathbb{C})$

In this section we provide a full description of the discrete subgroups in

$$\text{Heis}\,(3, \mathbb{C}) = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{C} \right\}.$$

We start with:

Proposition 5.1. *The whole group Heis $(3, \mathbb{C})$ is solvable and purely parabolic.*

This follows from the fact that every element in Heis $(3, \mathbb{C})$ has a lift to an upper diagonal matrix with all eigenvalues equal to 1. Then the proposition follows from the classification of the elements in PSL$(3, \mathbb{C})$, see [16, Ch. 4].

We split this section in four parts: in Subsection 5.1, for each discrete subgroup G in Heis $(3, \mathbb{C})$ we construct a region where a subgroup acts properly discontinuously; as a consequence we obtain Theorem 0.3 which is a decomposition theorem for G. In Subsection 5.2 we describe the subgroups of Heis $(3, \mathbb{C})$ for which the kernel of the control map, $Ker(\Pi|_G)$, is finite. The main tool here is the description of groups provided by Theorem 4.16. In 5.3 we describe the complex Kleinian groups in Heis $(3, \mathbb{C})$ with $Ker(\Pi|_G)$ infinite. Finally, in 5.4 we describe the discrete non-Kleinian groups in Heis $(3, \mathbb{C})$ with $Ker(\Pi|_G)$ infinite.

5.1. A discontinuity region for discrete subgroups of Heis $(3, \mathbb{C})$

Let us consider $G \subset \text{Heis}\,(3, \mathbb{C})$ a discrete group.

Definition 5.2. We set:

$$B(G) = \{(g_n) \subset G : (\Pi(g_n)) \text{ converges in PSL}(2, \mathbb{C})\};$$
$$L(G) = \{S \in \text{SP}(3, \mathbb{C}) : \text{ there is } (g_n) \in B(G) \text{ converging to } S\};$$
$$\mathcal{L}(G) = \{\ell \in \check{\mathbb{P}}^2_{\mathbb{C}} : \text{ there is } S \in L(G) \text{ satisfying } Ker(S) = \ell\};$$
$$\Omega(G) = \mathbb{P}^2_{\mathbb{C}} - \overline{\bigcup_{\ell \in \mathcal{L}(G)} \ell}.$$

Now we have:

Lemma 5.3. *For each $\ell \in \mathcal{L}(G)$ there exist a sequence $(g_n) \subset G$ of distinct elements and $P \in \text{SP}(3, \mathbb{C})$, such that the sequence $\Pi(g_n)$ converges to Id, (g_n) converges to P and $Ker(P) = \ell$.*

Proof. Let $\ell \in \mathcal{L}(G)$, then there exists a sequence $(h_n) \subset G$ of distinct elements and $P \in \text{SP}(3, \mathbb{C})$, such that $(\Pi(h_n))$ is a convergent sequence, h_n converges to P and $Ker(P) = \ell$. So we can assume that:

$$h_n = \begin{bmatrix} 1 & x_n & y_n \\ 0 & 1 & z_n \\ 0 & 0 & 1 \end{bmatrix} \text{ and } P = \begin{bmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Set $a_n = max\{|\,x_n\,|, |\,y_n\,|\}$, then it is not hard to check that there is a subsequence $(m_n) \subset (n)$ such that $a_n a_{m_n}^{-1}$ converges to 0 as n goes to ∞. Then

$$g_n = h_n^{-1} h_{m_n} = \begin{bmatrix} 1 & x_{m_n} - x_n & -y_n + x_n z_n + y_{m_n} - x_n z_{m_n} \\ 0 & 1 & z_{m_n} - z_n \\ 0 & 0 & 1 \end{bmatrix}.$$

Clearly $\Pi(g_n)$ converges to Id and $g_n \xrightarrow[n\to\infty]{} P$. $\square$

Lemma 5.4. *Let G be discrete and such that $Ker(\Pi|_G)$ is infinite and $\Pi(G)$ is non-trivial. If $\Omega(G)$ is non-empty, then:*

(1) *The group G acts properly discontinuously on $\Omega(G)$;*
(2) *The set $\Omega(G)$ is the largest open set on which G acts properly discontinuously.*
(3) *Each connected component of $\Omega(G)$ is homeomorphic to $\mathbb{R}^4$.*

Proof. We first prove (1). It is clear that $\Omega(G)$ is open and G-invariant; also, since $Ker(\Pi|_G)$ is infinite and $\Pi(G)$ is non-trivial, we have $\overleftrightarrow{e_1, e_2} \subset \mathbb{P}_{\mathbb{C}}^2 - \Omega(G)$. Now let $K \subset \Omega(G)$ be a compact set and $K(G) = \{g \in G : g(K) \cap K \neq \emptyset\}$. Assume that $K(G)$ is infinite. Let (g_n) be an enumeration of $K(G)$, then there exists a subsequence of (g_n), still denoted (g_n), such that either $(\Pi(g_n))$ converges to a projective transformation or $\Pi(g_n) \xrightarrow[n\to\infty]{} [e_2]$ uniformly on $\overleftrightarrow{e_2, e_3} - \{e_2\}$. If $(\Pi(g_n))$ converges to a projective transformation, we can find a subsequence $(h_n) \subset (g_n)$ and $\alpha \in L(G)$ such that h_n converges to α. Thus $Ker(\alpha) \in \mathcal{L}(G)$ and $Im(\alpha) = \{e_1\}$, therefore the accumulation set of $\{h_n(K) : n \in \mathbb{N}\}$ is $\{e_1\}$. Now, if $\Pi(g_n) \xrightarrow[n\to\infty]{} [e_2]$ uniformly on $\overleftrightarrow{e_2, e_3} - \{e_2\}$, then

$$\{(g_n) : n \in \mathbb{N}\} \subset \{g \in \Pi(G) : g(\pi(K)) \cap \pi(K) \neq \emptyset\},$$

which is not possible since $\overleftrightarrow{e_1, e_2} \subset \mathbb{P}_{\mathbb{C}}^2 - \Omega(G)$ and we have proved Part (1).

Now we prove (2). Let $\Omega \subset \mathbb{P}_{\mathbb{C}}^2$ be open, non-empty, G invariant and such that G acts properly discontinuously on Ω and $\ell \in \mathcal{L}(G)$. Then there are $(g_n) \in B(G)$ and $P \in L(G)$ such that $Ker(P) = \ell$ and (g_n) converges to P. By Lemma 5.3 we can assume that $\Pi(g_n)$ converges to Id. Proceeding as in Lemma 5.3, we conclude

$$g_n^{-1} = \begin{bmatrix} 1 & -x_n & x_n z_n - y_n \\ 0 & 1 & -z_n \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[n\to\infty]{} P.$$

By Lemma 1.14 we deduce $\ell \cap \Omega = \emptyset$.

Now we prove (3). If $\Pi(G)$ is discrete, then $\Omega(G)$ is $\Omega(Ker(\Pi|_G))$ by definition. From the definition of $\Omega(Ker(\Pi|_G))$ we get that $\Omega(Ker(\Pi|_G)) = Eq(Ker(\Pi|_G))$ and from

Example 2.3 we know that $Eq(Ker(\Pi|_G))$ is either $\mathbb{C}^2$ or $(\mathbb{C} \times \mathbb{H}^+) \cup (\mathbb{C} \times \mathbb{H}^-)$, proving the claim. So we now assume that $\Pi(G)$ is non-discrete; define

$$\mathcal{C}(G) = \overleftrightarrow{e_2, e_3} \cap \overline{\bigcup_{\ell \in \mathcal{L}(G)} \ell} \,,$$

then $\mathcal{C}(G)$ is a closed $\Pi(G)$-invariant set in $\overleftrightarrow{e_2, e_3}$, so $\mathcal{C}(G)$ is closed in $\mathbb{P}^2_{\mathbb{C}}$ and $\overline{\Pi(G)}$-invariant set. Since $\Pi(G)$ is non-discrete, there exists an additive Lie subgroup $H \subset \mathbb{C}$ such that $\overline{\Pi(G)} = \{z + b : b \in H\}$. Since $\mathcal{C}(G) - \{e_2\} = \overline{\Pi(G)}(\mathcal{C}(G) - \{e_2\})$, we deduce that $\mathbb{P}^2_{\mathbb{C}} - \Omega(G)$ is a pencil of lines over a union of real projective lines in $\overleftrightarrow{e_2, e_3}$, which are pairwise parallel in $\overleftrightarrow{e_2, e_3} - \{e_2\}$. Thus each connected component of $\Omega(G)$ is a fiber bundle over $\mathbb{R} \times \mathbb{R}$ with fiber $\mathbb{C}$, hence homeomorphic to $\mathbb{R}^4$. □

Definition 5.5. We say that the sequences $(a_n, b_n), (x_n, y_n) \subset \mathbb{C}^2$ are *co-bounded* if both sequences converge to ∞ and the sequence $\left(\frac{|a_n| + |b_n|}{|x_n| + |y_n|} \right)$ is bounded and bounded away from 0.

Lemma 5.6. *Let* $(a_n), (b_n), (c_n), (x_n), (y_n), (z_n) \subset \mathbb{C}$ *be sequences of distinct elements such that:*

(1) (c_n) *and* (z_n) *converge to* 0,
(2) $(a_n, b_n), (x_n, y_n)$ *are co-bounded;*
(3) $[a_n : b_n] \xrightarrow[n \to \infty]{} [a : b]$ *for some* a, b;
(4) $[x_n : y_n] \xrightarrow[n \to \infty]{} [x : y]$ *for some* x, y;
(5) $[a : b] \neq [x : y]$.

Then there exists $w \in \mathbb{C} \setminus \{0\}$ *such that for each* $k, m \in \mathbb{N} \setminus \{0\}$ *we get:*

$$g(n, k, m) = \begin{bmatrix} 1 & a_n & b_n \\ 0 & 1 & c_n \\ 0 & 0 & 1 \end{bmatrix}^k \begin{bmatrix} 1 & x_n & y_n \\ 0 & 1 & z_n \\ 0 & 0 & 1 \end{bmatrix}^m \xrightarrow[n \to \infty]{} \begin{bmatrix} 0 & ka + mxw & kb + myw \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Proof. Define $r_n = max\{|a_n|, |b_n|\}$, $s_n = max\{|x_n|, |y_n|\}$ and $t_n = max\{s_n, r_n\}$. Since $(a_n, b_n), (x_n, y_n)$ are co-bounded we can assume there are $r, s \in \mathbb{R} \setminus \{0\}$ such that $r_n t_n^{-1} \xrightarrow[n \to \infty]{} r$ and $s_n t_n^{-1} \xrightarrow[n \to \infty]{} s$. Moreover, since $[a_n : b_n] \xrightarrow[n \to \infty]{} [a : b]$ and $[x_n : y_n] \xrightarrow[n \to \infty]{} [x : y]$, we deduce that there are $u, v \in \mathbb{C}^*$ such that

$$r_n^{-1}(a_n, b_n) \xrightarrow[n \to \infty]{} u(a, b) \,,$$
$$s_n^{-1}(x_n, y_n) \xrightarrow[n \to \infty]{} v(x, y) \,.$$

Then an easy computation shows:

$$g(n,k,m) \xrightarrow[n\to\infty]{} \begin{bmatrix} 0 & ka+mxw & kb+myw \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $w = vs(ur)^{-1}$. $\quad\square$

Lemma 5.7. *Let* $G \subset \mathrm{Heis}(3,\mathbb{C})$ *be a Kleinian group such that* $\Pi(G)$ *is non-discrete and* $\mathbb{P}^2_{\mathbb{C}} - \Omega(G)$ *contains more than a line. Then* $\overline{\Pi(G)}$ *is isomorphic to* $\mathbb{R}$.

Proof. We have that $\Pi(G)$ is a non-discrete subgroup of $\mathbb{C}$, thus $\overline{\Pi(G)}$ must be isomorphic to $\mathbb{C}$, $\mathbb{R} \oplus \mathbb{Z}$ or $\mathbb{R}$, see Theorem 3.1 in [33]. Since G is complex Kleinian we deduce that $\overline{\Pi(G)}$ is isomorphic to either $\mathbb{R} \oplus \mathbb{Z}$ or $\mathbb{R}$. Let us assume that $\overline{\Pi(G)}$ is isomorphic to $\mathbb{R} \oplus \mathbb{Z}$. After conjugation, if necessary, we can assume that there exists $s > 0$ such that $\overline{\Pi(G)} = \{r + msi : r \in \mathbb{R}, m \in \mathbb{Z}\}$. Moreover, since $\mathcal{L}(G)$ contains more than a line, we can find a line $\ell \in \mathcal{L}(G)$ containing e_1 such that $\ell = \overleftrightarrow{e_1, [0:u:1]}$ where $Im(u) \neq 0$. On the other hand, by Lemma 5.3 we can find $(g_n) \subset G$ and $P \in \mathrm{SP}(3,\mathbb{C})$ such that $\Pi(g_n) \xrightarrow[n\to\infty]{} Id$, $g_n \xrightarrow[n\to\infty]{} P$ and $\ell = Ker(P)$. Thus there are sequences $(a_n), (b_n), (c_n) \subset \mathbb{C}$ such that $max\{|a_n|, |b_n|\} \xrightarrow[n\to\infty]{} \infty$, $c_n \xrightarrow[n\to\infty]{} 0$, $[a_n : b_n] \xrightarrow[n\to\infty]{} [a : b]$ and

$$g_n = \begin{bmatrix} 1 & a_n & b_n \\ 0 & 1 & c_n \\ 0 & 0 & 1 \end{bmatrix}; \quad P = \begin{bmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.1}$$

Thus $\ell = \overleftrightarrow{[0:b:-a], e_1}$ and $b = -ua$.

Claim 1. There are functions $f_2 : \mathbb{Z} \to \{\text{real projective subspaces of } \overleftrightarrow{e_2, e_3}\}$ and $f_1 : \mathbb{Z} \to \mathbb{C}$ such that:

(1) $Sgn(Im(f_1(m))) = Sgn(-m)$ for m large, here Sgn is the function sign;
(2) $|Im(f_1(m))| \xrightarrow[|m|\to\infty]{} \infty$;
(3) the point $[0 : b : -a]$ is in $\bigcap_{m\in\mathbb{Z}} f_2(m)$;
(4) for each $m \in \mathbb{Z}$ we have $[0 : f_1(m) : 1] \in f_2(m)$;
(5) $\bigcup_{m\in\mathbb{Z}} \bigcup_{p\in f_2(m)} \overleftrightarrow{e_1, p} \subset \mathbb{P}^2_{\mathbb{C}} \setminus \Omega(G)$.

Let $h \in G$ be of the form

$$h = \begin{bmatrix} 1 & x & y \\ 0 & 1 & is \\ 0 & 0 & 1 \end{bmatrix}, \text{ with } x, y \in \mathbb{C} \text{ and } s \in \mathbb{R}.$$

If g_n is given by Equation (5.1), then for each $m \in \mathbb{Z}$ we have:

$$h^{-m} g_n h^m = \begin{bmatrix} 1 & a_n & b_n - c_n mx + isma_n \\ 0 & 1 & c_n \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[n \to \infty]{} \begin{bmatrix} 0 & a & b + isma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

If for each $m \in \mathbb{Z}$ we apply Lemma 5.6 to the respective sequences induced by the sequences $(g_n)_{n \in \mathbb{N}}$ and $(h^{-m} g_n h^m)_{n \in \mathbb{N}}$ we deduce there exists $w_m \in \mathbb{C}^*$ such that:

$$C_m = \bigcup_{k,l \in \mathbb{Z}.} \overleftarrow{e_1, [0 : kbw_m + l(b + isma) : -kaw_m - la]} \subset \mathbb{P}^2_\mathbb{C} - \Omega(G) .$$

If for each $m \in \mathbb{Z}$ we define $g_2(m)$ as the closure of the set

$$[Span_\mathbb{Z}(\{w_m(b, -a), (b + isma, -a)\}) - \{0\}],$$

then by Lemma 5.11 in [3] we have that $g_2(m)$ is a real projective space that contains $[0 : b : -a]$. Now define $f_1(m) = u - ism$ and observe that $C_m = \bigcup_{p \in f_2(m)} \overrightarrow{e_1, p}$ and $[0 : u - isma : 1] \in f_2(m)$ for all $m \in \mathbb{Z}$.

To conclude the proof let f_1 and f_2 be the functions given above, then

$$G\left(\overline{\bigcup_{m \in \mathbb{Z}} \bigcup_{p \in f_2(m)} \overrightarrow{e_1, p}}\right) = \overline{\bigcup_{m \in \mathbb{Z}} \bigcup_{p \in \Pi(G) f_2(m)} \overrightarrow{e_1, p}} = \mathbb{P}^2_\mathbb{C} .$$

This yields $\Omega(G) = \emptyset$, which is a contradiction. $\square$

The proof of the following lemma is straightforward and it is left to the reader:

Lemma 5.8. *Set:*

$$\mathfrak{h}_\mathbb{C} = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{C} \right\} .$$

Then the map $\mathbf{exp} : \mathfrak{h}_\mathbb{C} \to \mathrm{Heis}(3, \mathbb{C})$, *given by*

$$\mathbf{exp} \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{bmatrix} 1 & a & b + 2^{-1}ac \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

is a diffeomorphism with inverse $\mathbf{log} : \mathrm{Heis}(3, \mathbb{C}) \to \mathfrak{h}_\mathbb{C}$ *given by*

$$\mathbf{log} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} 0 & a & b - 2^{-1}ac \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}.$$

Now we prove Theorem 0.3 stated in the introduction:

Proof. Let us show part (1). We know that G is finitely generated, therefore $\Pi(G)$ is finitely generated. If $n = rank(\Pi(G))$, let $H = \{g_1, \ldots, g_n\} \subset G$ be such that $\Pi(G)$ is generated by $\Pi(H)$. Let us consider the following equivalence relation in G: let us say that $a \sim b$ if and only if $[a, b] = Id$. If $A_1, \ldots, A_n$ are the equivalence classes in G induced by $\sim$, then define $B_0 = Ker(\Pi|_G))$ and $B_i = \langle A_i \rangle$. Now it is clear that $G = B_0 \rtimes \cdots \rtimes B_n$.

Now let us prove Part (2). Let $\{h_1, \ldots h_k\} \subset Ker(\Pi|_G)$ be a minimal generating set for $Ker(\Pi|_G)$ and let $\{g_1, \ldots, g_n\} \subset G$ be such that $\{\Pi(g_1), \ldots, \Pi(g_n)\}$ is a minimal generating set for $\Pi(G)$. Set

$$V = \left\{ \sum_{j=1}^{k} \alpha_j \mathbf{log}(h_j) + \sum_{j=1}^{n} \beta_j \mathbf{log}(g_j) : k_j, l_j \in \mathbb{Z} \right\}.$$

Claim 1. If $\mathfrak{h}_\mathbb{C}$ is as in Lemma 5.8, then V is an additive subgroup of $\mathfrak{h}_\mathbb{C}$ with rank $n + k$. For this, assume there are α_j, β_j's in $\mathbb{Z}$ such that

$$J = \sum_{j=1}^{k} \alpha_j \mathbf{log}(h_j) + \sum_{j=1}^{n} \beta_j \mathbf{log}(g_j) = 0.$$

We can assume that the g_j and h_j can be expressed in the following way:

$$h_j = \begin{bmatrix} 1 & u_j & v_j \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } g_j = \begin{bmatrix} 1 & x_j & y_j \\ 0 & 1 & z_j \\ 0 & 0 & 1 \end{bmatrix}.$$

Since the h_j generate $Ker(\Pi|_G)$ we have that $Span_\mathbb{Z}\{(u_j, v_j) : j = 1, \ldots, k\}$ is discrete; and since the $\Pi(g_j)$ generate $\Pi(G)$ we get that $\{z_1, \ldots z_n\}$ is a $\mathbb{Z}$-linearly independent set. An easy computation shows:

$$J = \begin{pmatrix} 0 & \sum_{j=1}^{k} \alpha_j u_j + \sum_{j=1}^{n} \beta_j x_j & \sum_{j=1}^{k} \alpha_j v_j + \sum_{j=1}^{n} \beta_j(y_j - 2^{-1} x_j z_j) \\ 0 & 0 & \sum_{j=1}^{n} \beta_j z_j \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $\mathbf{exp}(J) = Id$, we get the following system of equations:

$$\sum_{j=1}^{k} \alpha_j u_j + \sum_{j=1}^{n} \beta_j x_j = 0,$$

$$\sum_{j=1}^{n} \beta_j z_j = 0,$$

$$\sum_{j=1}^{k} \alpha_j v_j + \sum_{j=1}^{n} \beta_j(y_j - 2^{-1} x_j z_j) = 0.$$

Since $\{z_1, \ldots z_n\}$ is linearly independent over $\mathbb{Z}$ we conclude $\beta_1 = \ldots = \beta_n = 0$. Hence $\sum_{j=1}^{k} \alpha_j (u_j, v_j) = 0$ and therefore $\alpha_1 = \ldots = \alpha_k = 0$, proving the claim.

Let us define

$$\sqrt{[G,G]} = \{h \in \mathrm{Heis}\,(3, \mathbb{C}) : h^2 \in [G,G]\}.$$

It is clear that $\sqrt{[G,G]}$ is a discrete subgroup contained in the center of $\mathrm{Heis}\,(3, \mathbb{C})$.

Claim 2. $\left\langle G \cup \sqrt{[G,G]} \right\rangle$ is a discrete subgroup of $\mathrm{Heis}\,(3, \mathbb{C})$. Assume, on the contrary, that there exists a sequence $(f_n) \subset \left\langle G \cup \sqrt{[G,G]} \right\rangle$ of distinct elements such that $f_n \xrightarrow[n \to \infty]{} Id$, thus $f_n^2 \xrightarrow[n \to \infty]{} Id$. Since $(f_n^2) \subset G$ and G is discrete we deduce $f_n^2 = Id$ for n large, which is a contradiction.

Claim 3. $\log\left(\left\langle G \cup \sqrt{[G,G]} \right\rangle\right)$ is an additive discrete subgroup of $\mathfrak{h}_{\mathbb{C}}$. For this, let $a, b, c, x, y, z \in \mathbb{C}$ be such that:

$$\gamma_1 = \begin{pmatrix} 0 & a & b - 2^{-1}ac \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 & x & y - 2^{-1}xz \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \in \log\left(\left\langle G \cup \sqrt{[G,G]} \right\rangle\right).$$

An easy calculation shows:

$$\mathbf{exp}(\gamma_1 - \gamma_2) = \mathbf{exp}(\gamma_1)\mathbf{exp}(\gamma_2)^{-1} \begin{bmatrix} 1 & 0 & 2^{-1}(az - cx) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$\begin{pmatrix} 1 & 0 & 2^{-1}(az - cx) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 = [\mathbf{exp}(\gamma_1), \mathbf{exp}(\gamma_2)].$$

Hence $\mathbf{exp}(\gamma_1 - \gamma_2) \in \left\langle G \cup \sqrt{[G,G]} \right\rangle$. Since $\mathbf{exp}$ is a diffeomorphism with inverse $\mathbf{log}$, Claim 2 implies that $\mathbf{log}\left(\left\langle G \cup \sqrt{[G,G]} \right\rangle\right)$ is an additive discrete group.

To finish the proof of Part (2) we notice that V is a subgroup of the additive discrete group $\mathbf{log}\left(\left\langle G \cup \sqrt{[G,G]} \right\rangle\right) \subset \mathfrak{h}_{\mathbb{C}}$ and $dim_{\mathbb{R}}(\mathfrak{h}_{\mathbb{C}}) = 6$.

Now we prove Part (3). Let us assume that G is complex Kleinian. By Lemma 5.7, G leaves invariant each connected component of $\Omega(G)$, and each of these is contractible by Lemma 5.4. Hence, by Theorem 1.17, the obstruction dimension of G satisfies $obdim(G) \leq 4$. On the other hand each B_j is a finitely generated, torsion free group, and it is well known that this kind of groups are semi-hyperbolic, see [1]. Therefore Corollary 1.18 yields $\sum_{i=0}^{n} obdim(B_i) \leq obdim(G) \leq 4$. Finally notice that $B_j = \mathbb{Z}^{k_j}$ for some k_j and $obdim(B_j) = k_j = rank(B_j)$ by [11, 2.2]. $\quad\square$

Corollary 5.9. *If $G \subset \mathrm{Heis}(3, \mathbb{C})$ is a discrete group, then G is polycyclic.*

Recall that polycyclic means that the group is solvable and every subgroup is finitely generated. Polycyclic groups actually are finitely presented [23].

5.2. Triangular purely parabolic groups with trivial kernel

In this subsection we study purely parabolic groups with an invariant full flag and finite kernel. Now recall from Section 4.2 that U_+ is the subgroup of $\mathrm{PSL}(3, \mathbb{C})$ of classes of upper triangular matrices (g_{ij}) with $g_{11}g_{22}g_{33} = 1$, and we defined a group morphism $\Pi^* : U_+ \to Mob(\mathbb{C})$ by

$$\Pi^*([g_{ij}])z = g_{11}g_{22}^{-1}z + g_{12}g_{22}^{-1} .$$

Lemma 5.10. *Let $G \subset \mathrm{Heis}(3, \mathbb{C})$ be a commutative discrete group. If $Ker(\Pi^*|_G)$ and $Ker(\Pi|_G)$ are trivial, then there exist $W \subset \mathbb{C}$ an additive subgroup and $L : W \to \mathbb{C}$ a group morphism such that:*

(1) *The group G is conjugate to:*

$$\mathcal{K}_0(W, L) = \left\{ \begin{bmatrix} 1 & \xi & L(\xi) + 2^{-1}\xi^2 \\ 0 & 1 & \xi \\ 0 & 0 & 1 \end{bmatrix} : \xi \in W \right\}.$$

(2) *The Kulkarni limit set is:*

$$\Lambda_{Kul}(\mathcal{K}_0(W, L)) = \overleftrightarrow{e_1, e_2} = \mathbb{P}^2_{\mathbb{C}} - Eq(\mathcal{L}),$$

and its complement $\Lambda_{Kul}(\mathcal{K}_0(W, L))$ is the largest open set on which the group acts properly discontinuously.

(3) *The group $\mathcal{K}_0(W, L)$ is free with rank at most four.*

(4) *If W is discrete then L admits a linear extension to the real vector space $Span_{\mathbb{R}}(W)$.*

Proof. Let us show Part (1). Consider the following auxiliary function

$$\zeta : G \to \mathbb{C}^2$$
$$g \mapsto (\pi_{12}(g), \pi_{23}(g)) .$$

By definition ζ is a monomorphism. Set

$$\kappa : \zeta(G) \to \mathbb{C}$$
$$x \mapsto \pi_{13}(\zeta^{-1}(x)).$$

It is clear that we have:

$$G = \left\{ \begin{bmatrix} 1 & \pi_1(x) & \kappa(x) \\ 0 & 1 & \pi_2(x) \\ 0 & 0 & 1 \end{bmatrix} : x \in W \right\}.$$

Now let $x, y \in \zeta(G)$, then $A = B$ where A, B are:

$$A = \begin{bmatrix} 1 & \pi_1(x) & \kappa(x) \\ 0 & 1 & \pi_2(x) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \pi_1(y) & \kappa(y) \\ 0 & 1 & \pi_2(y) \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \pi_1(x+y) & \kappa(x) + \pi_1(x)\pi_2(y) + \kappa(y) \\ 0 & 1 & \pi_2(x+y) \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & \pi_1(y) & \kappa(y) \\ 0 & 1 & \pi_2(y) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \pi_1(x) & \kappa(x) \\ 0 & 1 & \pi_2(x) \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \pi_1(x+y) & \kappa(x) + \pi_1(y)\pi_2(x) + \kappa(y) \\ 0 & 1 & \pi_2(x+y) \\ 0 & 0 & 1 \end{bmatrix}$$

Then for every $x, y \in \zeta(G)$ we have:

$$\kappa(x+y) = \kappa(x) + \kappa(y) + \pi_1(x)\pi_2(y),$$

$$\pi_1(x)\pi_2(y) = \pi_1(y)\pi_2(x).$$

By Lemma 7.3 there exists an additive subgroup $W \subset \mathbb{C}$ and $\mu \in \mathbb{C}^*$ such that $\zeta(G) = W(1, \mu)$. Let us define

$$h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu^{-1/2} & 0 \\ 0 & 0 & \mu^{1/2} \end{bmatrix},$$

and observe that:

$$hGh^{-1} = \left\{ \begin{bmatrix} 1 & \xi & \widetilde{\kappa}(\xi) \\ 0 & 1 & \xi \\ 0 & 0 & 1 \end{bmatrix} : \xi \in W \right\},$$

where $\widetilde{\kappa} : W \to \mathbb{C}$ satisfies $\widetilde{\kappa}(\xi_1 + \xi_2) = \widetilde{\kappa}(\xi_1) + \widetilde{\kappa}(\xi_2) + \xi_1 \xi_2$. To conclude define $L : W \to \mathbb{C}$ by $L(\xi) = \widetilde{\kappa}(\xi) - 2^{-1}\xi^2$, then:

$$L(\xi_1 + \xi_2) = \widetilde{\kappa}(\xi_1 + \xi_2) - 2^{-1}(\xi_1 + \xi_2)^2 = \widetilde{\kappa}(\xi_1) + \widetilde{\kappa}(\xi_2) - 2^{-1}\xi_1^2 - 2^{-1}\xi_2^2 = L(\xi_1) + L(\xi_2),$$

proving Part (1).

Let us prove Part (2). Let $(g_m) \subset G$ be a sequence of distinct elements of G, then there exists $(x_n) \subset W$ a sequence of distinct elements such that

$$g_m = \begin{bmatrix} k_m^{-1} & x_m k_m^{-1} & k_m^{-1}(L(x_m) + x^2/2) \\ 0 & k_m^{-1} & x_m k_m^{-1} \\ 0 & 0 & k_m^{-1} \end{bmatrix},$$

where $k_m = max\{|x_m|, |L(x_m) + x_m^2/2|\}$. If (g_{n_m}) is a subsequence of (g_m) such that (g_{n_m}) converges to $P \in \mathrm{SP}(3, \mathbb{C}) - \mathrm{PSL}(3, \mathbb{C})$, then there are $a, b \in \mathbb{C}$ such that $| a | + | b | \neq 0$ and

$$P = \begin{bmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix}.$$

This shows that $Eq(G) = \mathbb{P}^2_{\mathbb{C}} - \overleftrightarrow{e_1, e_2} = \mathbb{C}^2$ since, by Proposition 1.6, $Eq(G) \subset \Omega_{Kul}(G)$ and $\Lambda_{Kul}(G)$ always contains a line. Then $Eq(G) = \Omega_{Kul}(G)$. If $\Omega \subset \mathbb{P}^2_{\mathbb{C}}$ is any open set on which G acts properly discontinuously, then $\mathbb{P}^2_{\mathbb{C}} - \Omega$ contains a complex line, say ℓ. If $g \in G - \{Id\}$, then $g^m \ell \xrightarrow[m \to \infty]{} \overleftrightarrow{e_1, e_2}$.

In order to prove Part (3) we observe that G is an group acting properly discontinuously and freely on $\mathbb{C}^2$, thus the rank of G must be at most four, see the proof of Proposition 5.9 in [4]. The last part of the theorem is immediate. $\square$

As a consequence of Lemma 5.10 in [3] we get the following result.

Lemma 5.11. *Let $G \subset \mathrm{Heis}(3, \mathbb{C})$ be a commutative discrete group, then:*

(1) *If $Ker(\Pi^*|_G)$ is non-trivial, then there is a discrete additive subgroup $\mathcal{L}$ of $\mathbb{C}^2$ with rank at most four, such that:*

$$G = \left\{ \begin{bmatrix} 1 & 0 & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} : (y, z) \in \mathcal{L} \right\}.$$

(2) *If $Ker(\Pi|_G)$ is non-trivial, then there exists a discrete additive subgroup $\mathcal{L} \subset \mathbb{C}^2$ such that:*

$$G = \left\{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : x \in \mathcal{L} \right\}.$$

Moreover, if G is complex Kleinian, then $\mathcal{L}$ has rank at most 2.

5.3. Groups with infinite Kernel

We consider now discrete groups $G \subset \mathrm{Heis}(3, \mathbb{C})$ whose control map has infinite kernel, *i.e.*, $Ker(\Pi|_G)$ is infinite.

Lemma 5.12. *If G is complex Kleinian group with infinite kernel, then:*

(1) We have that $Ker(\Pi|_G) = \mathbb{Z}^k$ where $1 \leq k \leq 2$.

(2) We have that $\Lambda_{Kul}(Ker(\Pi|_G)) = L_0(Ker(\Pi|_G))$ is either a line or a pencil of lines over a circle, where L_0 is as in Definition 1.5.

(3) If the set $\Lambda_{Kul}(Ker(\Pi|_G))$ is a line, then there exists a discrete additive subgroup W of $\mathbb{C}$ such that G is conjugate to:

$$G_W = \left\{ \begin{bmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : w \in W \right\}$$

and $rank(Ker(\Pi|_G)) \leq 2$.

(4) If $\Lambda_{Kul}(Ker(\Pi|_G))$ is a pencil of lines over a circle, then the rank of $Ker(\Pi|_G)$ is two and the groups $\Pi^*(Ker(\Pi|_G))$ and $\pi_{23}(Ker(\Pi|_G))$ are non-trivial.

(5) If the group $\Pi(G)$ is non-trivial. Then the group $\Pi^*(Ker(\Pi|_G))$ is non-trivial if and only if $\Lambda_{Kul}(Ker(\Pi|_G))$ is a pencil of lines over a circle.

Proof. The proofs of parts (1) and (2) follow from Example 2.3. Let us prove Part (3). It is clear that there exists $\mathcal{L} \subset \mathbb{C}^2$ an $\mathbb{R}$-linearly independent set such that $G = \mathcal{T}^*(\mathcal{L})$, where $G = \mathcal{T}^*(\mathcal{L})$ is given as in Example 2.3. We know that

$$\Lambda_{Kul}(Ker(\Pi|_G)) = \bigcup_{p \in S} \overleftrightarrow{e_1, p},$$

where S is the closure of the set $\{Span_{\mathbb{Z}}\{(y, -x) : (x, y) \in \mathcal{L}\} \setminus \{0\}\}$. Since $\Lambda_{Kul}(Ker(\Pi|_G))$ is a single line, from Lemma 2.3 we deduce that S is either a single point or it contains exactly two $\mathbb{C}$-linearly dependent vectors. Let us assume that S contains exactly two $\mathbb{C}$-linearly dependent vectors, the other case is similar; so there exists $\alpha \in \mathbb{C}$ and $(x, y) \in \mathcal{L}$, such that one has:

$$G = \left\{ \begin{bmatrix} 1 & (n+m\alpha)x & (n+m\alpha)y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : n, m \in \mathbb{Z} \right\}.$$

Let $r \in \mathbb{R}^*$ be such that $x \neq yr$, then a simple computation shows:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r & 1 \\ 0 & x & y \end{bmatrix} \begin{bmatrix} 1 & (n+m\alpha)x & (n+m\alpha)y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & r & 1 \\ 0 & x & y \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & m+n\alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

proving (3). Notice that Part (4) follows from Example 2.3, so let us prove (5).

Since $\Pi^*(Ker(\Pi|_G))$ and $\Pi(G)$ are both non-trivial, we deduce that there are $a, b, x, y, z \in \mathbb{C}$ and $g, h \in G$ such that $az \neq 0$ and

$$g = \begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad h = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}.$$

By a straightforward computation we find:

$$hgh^{-1} = \begin{bmatrix} 1 & a & b - az \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In order to conclude the proof we only need to observe that (a, b) and $(a, b - az)$ are $\mathbb{C}$-linearly independent vectors. $\square$

As in [3], we use the notation $\mu(U)$ to denote the maximum number of complex projective lines in general position contained in $\mathbb{P}^2_{\mathbb{C}} - U$.

Lemma 5.13. *Let $G \subset \mathrm{Heis}(3, \mathbb{C})$ be complex Kleinian group such that $\Pi(G)$ is non trivial and $\mu(\Omega_{Kul}(Ker(\Pi|_G))) = 2$, then*

(1) *The group $\Pi(G)$ is discrete.*
(2) *The rank of $\Pi(G)$ is equal to one.*

Proof. Assume $\Pi(G)$ is not discrete. Then we can assume there exists a sequence $(g_n) \subset G$ such that $\Pi(g_n)$ is a sequence of distinct elements converging to Id. On the other hand, since $\Lambda_{Kul}(Ker(\Pi|_G))$ is a pencil of lines over a circle, there exists $g \in Ker(\Pi|_G)$ such that $\Pi^*(g) \neq Id$. If g_n and g are given respectively by

$$g_n = \begin{bmatrix} 1 & a_n & b_n \\ 0 & 1 & c_n \\ 0 & 0 & 1 \end{bmatrix} ; \quad g = \begin{bmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then

$$g_n g g_n^{-1} = \begin{bmatrix} 1 & x & y - xc_n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[n \to \infty]{} \begin{bmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which contradicts that G is discrete.

Now we assume that $\Pi(G)$ has rank ≥ 2. Let $h_1, h_2, h \in G$ be such that $\langle \Pi(h_1), \Pi(h_2) \rangle = \Pi(G)$, $h \in Ker(\Pi|_G)$ and $\Pi^*(h) \neq Id$. Set:

$$h_1 = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} ; \quad h_2 = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} ; \quad h = \begin{bmatrix} 1 & u & v \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then

$$[h^{-1}, h_1] = \begin{bmatrix} 1 & 0 & -uc \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \quad [h^{-1}, h_2] = \begin{bmatrix} 1 & 0 & -uz \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $\{(u,v),(0,-uc),(0,-uz)\}$ is an $\mathbb{R}$-linearly independent set, which is not possible. $\square$

Proposition 5.14. *If $G \subset \mathrm{Heis}(3,\mathbb{C})$ is complex Kleinian such that $\Pi(G)$ is non trivial and $\mu(\Omega_{Kul}(Ker\Pi|_G)) = 2$. Then there exist $x, y \in \mathbb{C}$, $p, q, r \in \mathbb{Z}$ such that p, q are co-primes, q^2 divides r and G is conjugate to*

$$H = \left\{ \begin{bmatrix} 1 & k + lpq^{-1} + mx & lr^{-1} + m\left(k + lpq^{-1}\right) + \binom{m}{2}x + my \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix} : (k,l,m) \in \mathbb{Z} \right\}.$$

Proof. By Lemma 5.13 we know that $\Pi(G)$ is discrete and has rank equal to 1; and by Lemma 5.12 we have $rank(Ker(\Pi|_G)) = 2$ and $\Pi^*(Ker(\Pi|_G))$ is non-trivial. Thus by Theorem 0.3 there exist $\{(a,b),(c,d)\}$ a $\mathbb{C}$-linearly independent set and $u, v, w \in \mathbb{C}$ such that

$$G = \left\{ \begin{bmatrix} 1 & ka + lc & kb + ld \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & u & v \\ 0 & 1 & w \\ 0 & 0 & 1 \end{bmatrix}^n : k, l, n \in \mathbb{Z} \right\}, \tag{5.2}$$

$aw \neq 0$. A simple computation shows:

$$\begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & 1 & \frac{b}{a} \\ 0 & 0 & w \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & 1 & \frac{b}{a} \\ 0 & 0 & w \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = g_1,$$

$$\begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & 1 & \frac{b}{a} \\ 0 & 0 & w \end{bmatrix} \begin{bmatrix} 1 & c & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & 1 & \frac{b}{a} \\ 0 & 0 & w \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \frac{c}{a} & \frac{d}{aw} - \frac{bc}{a^2 w} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = g_2,$$

$$\begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & 1 & \frac{b}{a} \\ 0 & 0 & w \end{bmatrix} \begin{bmatrix} 1 & u & v \\ 0 & 1 & w \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & 1 & \frac{b}{a} \\ 0 & 0 & w \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \frac{u}{a} & \frac{v}{aw} - \frac{bu}{awa} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = g_3.$$

Now by Equation (5.2) we deduce that

$$G_1 = \{g_1^k g_2^l g_3^n : k, l, n \in \mathbb{Z}\}$$

is a group conjugate to G. On the other hand, G_1 is a group if and only if

$$g_3 g_i g_3^{-1} \in \langle g_1, g_2 \rangle \text{ for } i = 1, 2.$$

The last statement is equivalent to

$$(0,1), (0, ca^{-1}) \in Span_{\mathbb{Z}}(\{(1,0), (ca^{-1}, (aw)^{-1}(d - bca^{-1}))\}).$$

Now the conclusion follows from Lemma 7.4. $\square$

Proposition 5.15. *Let $G \subset \mathrm{Heis}(3, \mathbb{C})$ be complex Kleinian such that $\Lambda_{Kul}(Ker(\Pi|_G))$ is a line and $\Pi(G)$ is discrete. Then G is conjugate to one of the following groups:*

(1)

$$
\mathcal{T}(\mathcal{L}) = \left\{ \begin{bmatrix} 1 & 0 & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} : (y, z) \in \mathcal{L} \right\}, \tag{5.3}
$$

where $\mathcal{L} \subset \mathbb{C}^2$ is an additive subgroup such that $\pi_2(\mathcal{L})$ is discrete.

(2)

$$
\mathcal{K}_0(W_1, W_2, L) = \left\{ \begin{bmatrix} 1 & x & L(x) + x^2/2 + w \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} : w \in W_2, x \in W_1 \right\}, \tag{5.4}
$$

where $W_1, W_2 \subset \mathbb{C}$ are additive discrete subgroups and $L : W_1 \to \mathbb{C}$ is a group morphism.

(3)

$$
\mathcal{K} = \left\{ \begin{bmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & x \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^n \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}^m : m, n \in \mathbb{Z}, w \in W \right\}, \tag{5.5}
$$

where $W \subset \mathbb{C}$ is an additive discrete subgroup, $a - c \in W$ and $c \notin \mathbb{R}$.

Proof. Since $\Pi(G)$ is discrete we deduce $Ker(\Pi|_G)$ and $\Pi(G)$ are torsion free Abelian groups with rank less than or equal to 2. For simplicity we may assume that $rank(Ker(\Pi|_G)) = rank(\Pi(G)) = 2$ since, as we will see in the proof, any other possibility will be covered by this case. Now by Theorem 0.3 there exist $W \subset \mathbb{C}$ an additive discrete subgroup with rank 2 and $a, b, c, x, y, z \in \mathbb{C}$ such that:

$$
G = \left\{ \begin{bmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}^m \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}^n : w \in W, m, n \in \mathbb{Z} \right\}, \tag{5.6}
$$

and $zc^{-1} \notin \mathbb{R}$. Now consider the following cases:

Case 1. $xb - za = 0$. Let us consider the following sub-cases:

Sub-case 1. $x = a = 0$. Then from Equation (5.6) we see that G is conjugate to the torus group given by Equation (5.3).

Sub-case 2. $xa \neq 0$. Observe that:

$$
g_w = \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z \end{bmatrix} \begin{bmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & w(xz)^{-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
$$

$$g = \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z \end{bmatrix} \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & \frac{y}{xz} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$h = \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \frac{a}{x} & \frac{b}{xz} \\ 0 & 1 & \frac{c}{x} \\ 0 & 0 & 1 \end{bmatrix}.$$

By Lemma 5.10 there exists a group morphism $L : W_1 = Span_{\mathbb{Z}}(\{1, ax^{-1}\}) \to \mathbb{C}$ such that:

$$\langle h, g \rangle = \left\{ \begin{bmatrix} 1 & r & L(r) + 2^{-1}r^2 \\ 0 & 1 & r \\ 0 & 0 & 1 \end{bmatrix} : r \in W_1 \right\}.$$

Now by Equation (5.6) we have that G is conjugate to an Abelian Kodaira group as in Equation (5.4).

From Lemma 5.12 it is clear that the previous cases cover all the possibilities for the case $xb - za = 0$.

Case 2. $xb - za \neq 0$.

Sub-case 1. $x = 0$, $a \neq 0$. The following equations:

$$g_w = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z \end{bmatrix} \begin{bmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & w(az)^{-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$g = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z \end{bmatrix} \begin{bmatrix} 1 & 0 & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & \frac{y}{az} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$h = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & \frac{b}{az} \\ 0 & 1 & \frac{c}{z} \\ 0 & 0 & 1 \end{bmatrix},$$

together with Equation (5.6), imply that G is conjugate to a group of the form given by Equation (5.5).

Sub-case 2. $x \neq 0$ and $a \neq 0$. Then analogous arguments show that G is conjugate to a group of the form given by Equation (5.5). □

In a similar way one can show the following proposition:

Proposition 5.16. *Let $G \subset \mathrm{Heis}(3, \mathbb{C})$ be a commutative complex Kleinian group such that $Ker(\Pi|_G)$ is infinite and $\Pi(G)$ is non-discrete, then:*

(1) *If $\Pi^*(G)$ is trivial, then there exists $\mathcal{L} \subset \mathbb{C}^2$ an additive discrete subgroup of rank at most four, such that $\pi_2(\mathcal{L})$ is non-discrete and G is conjugate to:*

$$\mathcal{T}(\mathcal{L}) = \left\{ \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} : (a,b) \in \mathcal{L} \right\}.$$

(2) *If $\Pi^*(G)$ is non trivial, then there exist additive subgroups $W_1, W_2 \subset \mathbb{C}$ such that W_1 is non-discrete, W_2 has rank 1 and G is conjugate to:*

$$\mathcal{K}_0(W_1, W_2, L) = \left\{ \begin{bmatrix} 1 & x & L(x) + x^2/2 + w \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} : w \in Span_{\mathbb{Z}}(W_2), \ x \in Span_{\mathbb{Z}}(W_1) \right\}$$

where $L : W_1 \to \mathbb{C}$ is a group morphism.

Lemma 5.17. *Let $a, b, c \in \mathbb{C}$ and $r \in \mathbb{R} - \mathbb{Q}$, and let $G \subset \mathrm{Heis}(3, \mathbb{C})$ be the group given by*

$$G = \left\langle A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & a+r & b \\ 0 & 1 & r \\ 0 & 0 & 1 \end{bmatrix} \right\rangle.$$

Then

(1) *G is commutative if and only if $a = 0$.*
(2) *If $a \neq 0$, then $\mathbb{P}^2_{\mathbb{C}} - \Omega(G)$ is a cone of lines over a circle.*

Proof. The proof of Part (1) is straightforward. To prove Part (2) notice that Theorem 0.3 implies:

$$G = \left\{ g_{mnk} = \begin{bmatrix} 1 & (m+nr)+na & ak+\binom{m}{2}+mnr+\binom{n}{2}r(r+a) \\ 0 & 1 & m+nr \\ 0 & 0 & 1 \end{bmatrix} : k, m, n \in \mathbb{Z} \right\}$$

By elementary algebra we have:

$$\frac{a(2k + n^2 r) + rn(-a - r + 1) + (m + nr)^2 - (m + nr)}{2}$$

$$= ak + \binom{m}{2} + mnr + \binom{n}{2} r(r + a). \tag{5.7}$$

Claim 1. $\mathbb{P}^2_{\mathbb{C}} - \Omega(G)$ contains more than one line. To prove this claim it is enough to show that $\mathbb{P}^2_{\mathbb{C}} - \Omega(G)$ contains a line different from $\overleftrightarrow{e_1, e_2}$. Let $(a_n), (b_n) \in \mathbb{Z}$ be sequences such that $a_n + b_n r \xrightarrow[n \to \infty]{} 0$; let us assume that all the elements in the sequence (a_n) are either odd or even. Let $k_0 \in \mathbb{N}$ be an even number such that

$$k_0|a| > |r(-a - r + 1)|,$$

and define the following sequence:

$$c_n = \begin{cases} 2^{-1}b_n(a_n + k_0 + 1) & \text{if } a_n \text{ is odd}, \\ 2^{-1}b_n(a_n + k_0) & \text{if } a_n \text{ is even}. \end{cases}$$

Clearly $(c_n) \subset \mathbb{Z}$ and

$$g_{a_n,b_n,c_n} \xrightarrow[n\to\infty]{} g = \begin{bmatrix} 0 & 2a & w_0 + r(-a - r + 1) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where w_0 is either k_0 or $k_0 + 1$. Hence $Ker(g)$ is a complex line distinct from $\overleftrightarrow{e_1, e_2}$.

Claim 2. $\mathbb{P}^2_{\mathbb{C}} - \Omega(G)$ is contained in a pencil of lines over an Euclidean circle. Let $(a_n), (b_n), (c_n) \in \mathbb{Z}$ be sequences such that $a_n + b_n r \xrightarrow[n\to\infty]{} 0$. Assume that

$$g_{a_n,b_n,c_n} \xrightarrow[n\to\infty]{} g = \begin{bmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

If $x \neq 0$ we get:

$$
\begin{aligned}
x &= \lim_{n\to\infty} 2(a_n + b_n r + b_n a)b_n^{-1} = 2a, \\
y &= \lim_{n\to\infty}(a(2c_n + b_n^2 r) + rb_n(-a - r + 1) + (a_n + b_n r)^2 - (a_n + b_n r))b_n^{-1} \\
&= r(-a - r + 1) + a \lim_{n\to\infty}(2c_n b_n^{-1} + b_n r).
\end{aligned}
$$

Thus $y = sa + 1 - r$ for some $s \in \mathbb{R}$. Therefore:

$$\mathbb{P}^2_{\mathbb{C}} - \Omega(G) \subset \overleftrightarrow{e_1, e_2} \cup \bigcup_{s\in\mathbb{R}} \overleftrightarrow{e_1, [0 : sa + 1 - r : -2a]}.$$

Finally, since $\Pi(G)$ is conjugate to a dense subgroup of $\mathbb{R}$ and $\mathbb{P}^2_{\mathbb{C}} - \Omega(G)$ has more than two lines we deduce $\mathbb{P}^2_{\mathbb{C}} - \Omega(G)$ contains a pencil of lines over an Euclidean circle. □

Lemma 5.18. *Let $G \subset \mathrm{Heis}(3, \mathbb{C})$ be a non-Abelian Kleinian group such that $Ker(\Pi|_G)$ is infinite and $\Pi(G)$ is non-discrete, then:*

(1) *The set $\Lambda_{Kul}(Ker(\Pi|_G))$ is a complex line;*
(2) *The set $\mathbb{P}^2_{\mathbb{C}} - \Omega(G)$ contains more than one line;*
(3) *The group $\Pi(G)$ is conjugate to a subgroup of $\mathbb{R}$;*
(4) *The rank of the group $\Pi(G)$ is equal to two.*

Proof. Part (1) follows from Lemma 5.13. Let us prove (2). Since G is non-commutative, there are $x, y, z, a, b, c \in \mathbb{C}$ such that $\{z, c\}$ is $\mathbb{R}$-linearly dependent but it is a $\mathbb{Z}$-linearly independent set and also: $xc - az \neq 0$ and

$$g = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}, h = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \in G.$$

Since $[g, h] \neq Id$ we can assume $x \neq 0$. A simple computation shows:

$$g_1 = \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & 1 & \frac{y}{x} \\ 0 & 0 & z \end{bmatrix} \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & 1 & \frac{y}{x} \\ 0 & 0 & z \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$h_1 = \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & 1 & \frac{y}{x} \\ 0 & 0 & z \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & 1 & \frac{y}{x} \\ 0 & 0 & z \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \frac{a}{x} & \frac{b}{xz} - \frac{ay}{x^2 z} \\ 0 & 1 & \frac{c}{z} \\ 0 & 0 & 1 \end{bmatrix}.$$

Then Part (2) follows by applying Lemma 5.17 to the group $\langle g_1, h_1 \rangle$.

The proof of (3) is immediate from Lemma 5.7, so let us prove Part (4). Assume that G has a non-commutative subgroup H of type $\mathcal{K}_4$ such that $\Pi(H)$ is non-discrete and has rank 3. Since H is not commutative, by Theorem 0.3, the previous parts of this lemma and after conjugation, if necessary, we can find an additive discrete subgroup $W \subset \mathbb{C}$, $a, b, c, r, s, t \in \mathbb{C}$ such that $a \neq 0$, $\{1, t, c\}$ is $\mathbb{R}$-linearly dependent but $\mathbb{Z}$-linearly independent and:

$$H = \left\{ \begin{bmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^n \begin{bmatrix} 1 & a+c & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}^m \begin{bmatrix} 1 & r+t & s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}^k : k, m, n \in \mathbb{Z}, w \in W \right\}.$$

Since H is a group, this means $a, r, rc - at \in W$. By Kronecker Theorem [32, Theorem 4.1], W is non-discrete, which is a contradiction. $\square$

The proof of the following proposition is left to the reader.

Proposition 5.19. *Let $G \subset \mathrm{Heis}(3, \mathbb{C})$ be a non-Abelian complex Kleinian group such that $Ker(\Pi|_G)$ is infinite and $\Pi(G)$ is non-discrete. Then there are a rank one additive discrete subgroup $W \subset \mathbb{C}$, $a \in W$, and $b, c \in \mathbb{C}$, such that $\{1, c\}$ is $\mathbb{R}$-linearly dependent but $\mathbb{Z}$-linearly independent and up to conjugation we have:*

$$G = \left\{ \begin{bmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^n \begin{bmatrix} 1 & a+c & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}^m : m, n \in \mathbb{Z}, w \in W \right\}.$$

Now we prove

Theorem 5.20. *Let G be a purely parabolic discrete group in* $\mathrm{PSL}(3, \mathbb{C})$. *Then G is virtually finitely presented, torsion free and solvable. Also, G is virtually either unipotent and conjugate to the projectivization of a subgroup of* $\mathrm{Heis}(3, \mathbb{C})$, *or else it is an Abelian group of rank at most two, with an irrational ellipto-parabolic element, and it is of the form:*

$$Ell(W, \mu) = \left\{ \begin{bmatrix} \mu(w) & \mu(w)w & 0 \\ 0 & \mu(w) & 0 \\ 0 & 0 & \mu(w)^{-2} \end{bmatrix} : w \in W \right\},$$

where W is a discrete additive subgroup of $\mathbb{C}$ and $\mu : W \to \mathbb{S}^1$ is a group morphism.

Proof. Let G be a discrete group in $\mathrm{PSL}(3, \mathbb{C})$ with no loxodromic elements. By Theorem 4.3 we have that G contains a finite index subgroup which is conjugate to a group G_0 which is the projectivization of an upper triangular group of matrices. Then Lemma 4.10 grants the existence of a finite index, torsion free subgroup G_1 of G_0, for which the following groups are all torsion free: $\Pi(G_1)$, $\Pi^*(G_1)$, $\lambda_{12}(G_1)$, $\lambda_{13}(G_1)$ and $\lambda_{23}(G_1)$. Now use Theorems 4.3 and 4.5 applied to G_1 and deduce that either G_1 is a unipotent subgroup of $\mathrm{Heis}\,(3, \mathbb{C})$ or else it is Abelian of rank at most 2 of the form stated in Theorem 5.20. Now, if G_2 is a discrete subgroup of $\mathrm{Heis}\,(3, \mathbb{C})$ then by Corollary 5.9 we have that G_2 is solvable and finitely presented. $\quad\square$

Finally we have:

Proof of Theorem 3.1. Let G_1 be a subgroup of G which is triangularizable. Let $G_0 \subset G_1$ be a subgroup of finite index, such that $G_0, \lambda_{12}(G_0), \lambda_{23}(G_0), \Pi(G_0), Ker(\Pi|_{G_0})$ are torsion free, see Lemma 4.10. If G_0 contains a parabolic element g satisfying

$$max\{o(\lambda_{12}(g)), o(\lambda_{23}(g))\} = \infty,$$

then the result follows from Theorem 4.5. Thus we can assume that $G \subset \mathrm{Heis}\,(3, \mathbb{C})$. If $Ker(\Pi|_{G_0})$ is trivial we deduce that G_0 is commutative. The proof in this case follows from Theorem 4.16 and Lemmas 4.17, 5.10. If $Ker(\Pi|_{G_0})$ is non-trivial the result follows from Lemma 5.12 and Propositions 5.14, 5.15, 5.16 and 5.19. $\quad\square$

5.4. Discrete groups of $Heis\,(3, \mathbb{C})$ which are not complex Kleinian

Proposition 5.21. *Let $G \subset \mathrm{Heis}(3, \mathbb{C})$, then G is a discrete non-commutative group such that $\Pi(G)$ is discrete and non-trivial if and only if G is conjugate to either*

$$\mathcal{K} = \left\{ \begin{bmatrix} 1 & u & v \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & x \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^n \begin{bmatrix} 1 & a+c & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}^m : m, n \in \mathbb{Z}, (u, v) \in \mathcal{L} \right\},$$

or

$$W_{x,a,b} = \left\{ \begin{bmatrix} 1 & u & v \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & x \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^n : n \in \mathbb{Z}, (u,v) \in W \right\},$$

where $a, b, c, x \in \mathbb{C}$, $c \notin \mathbb{R}$ and $\mathcal{L} \subset \mathbb{C}^2$ is an additive discrete subgroup satisfying that $Span_{\mathbb{Z}}\{(0,a), (0, \pi_1(\mathcal{L})), (0, c\pi_1(\mathcal{L}))\} \subset \mathcal{L}$ and $rank(\mathcal{L}) \geq 3$.

Proof. Let us assume that G is a discrete non-commutative group such that $\Pi(G)$ is discrete and non-trivial. Without loss of generality let us assume that $\Pi(G)$ has rank two. Then by Theorem 0.3 there are $a, b, c, x \in \mathbb{C}$, $c \notin \mathbb{R}$ and $\mathcal{L} \subset \mathbb{C}^2$ is an additive discrete subgroup such that G is conjugate to the group:

$$\left\{ \begin{bmatrix} 1 & u & v \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & x \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^n \begin{bmatrix} 1 & a+c & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}^m : m, n \in \mathbb{Z}, (u,v) \in \mathcal{L} \right\}.$$

It is clear that G acts properly discontinuously on $\Omega(Ker(\Pi|_G))$. Since G is not complex Kleinian we deduce $rank(\mathcal{L}) \geq 3$. For $w = (u,v) \in \mathcal{L}$ and $k, l, m \in \mathbb{Z}$, define:

$$g(w,k,l,m) = \begin{bmatrix} 1 & u & v \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & x \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^k \begin{bmatrix} 1 & a+c & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}^l.$$

Let $w_i = (u_i, v_i) \in \mathcal{L}$ $(i = 1, 2)$ and $k, l, m, n \in \mathbb{Z}$; a straightforward computation shows:

$$g(w_1, k, l,)g(w_2, m, n)^{-1} = g(w_1 - w_2 + w, k - m, l - n)$$

where $w = u_2(cl - cn + k - m) - ma(l - n)$. Thus

$$Span_{\mathbb{Z}}\{(0,a), (0, \pi_1(\mathcal{L})), (0, c\pi_1(\mathcal{L}))\} \subset \mathcal{L}. \quad \square$$

The proof of the following lemma is a slight modification of the proof of Part (2) in Lemma 5.13, so we omit it.

Lemma 5.22. *Let $G \subset \mathrm{Heis}(3, \mathbb{C})$ be a discrete non-commutative group such that $\Pi(G)$ is non-discrete. Then $\Lambda_{Kul}(Ker(\Pi|_G))$ is a single complex projective line.*

The next result is a direct consequence of Proposition 5.19 and we include it without proof:

Proposition 5.23. *Let $G \subset \mathrm{Heis}(3, \mathbb{C})$ be a non-Abelian discrete but not Kleinian group such that $Ker(\Pi|_G)$ is infinite and $\Pi(G)$ is a rank two non-discrete group. Then we can*

find a rank two additive discrete subgroup $W \subset \mathbb{C}$, $a \in W$, $b, c \in \mathbb{C}$ such that $\{1, c\}$ is $\mathbb{R}$-linearly dependent but $\mathbb{Z}$-linearly independent and up to conjugation we have:

$$G = \left\{ \begin{bmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^n \begin{bmatrix} 1 & a+c & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}^m : m, n \in \mathbb{Z}, w \in W \right\}.$$

Lemma 5.24. *Let* $G \subset \mathrm{Heis}(3, \mathbb{C})$ *be a discrete non-commutative group such that* $\Pi(G)$ *has rank at least 3, then:*

(1) *For every 1-dimensional real subspace* $\ell \subset \mathbb{C}$ *we have* $rank(\ell \cap \Pi(G)) \leq 2$.
(2) *We have* $rank(Ker(\Pi|_G)) = 2$ *and* $3 \leq rank(\Pi(G)) \leq 4$.

Proof. Let us prove Part (1). Assume there is a real line $\ell \subset \mathbb{C}$ for which $rank(\ell \cap \Pi(G)) \geq 3$. Then there are $x \in \mathbb{C}^*$ and $r, s \in \mathbb{R}^*$ such that $Span_{\mathbb{Z}}\{1, r, s\}$ is a rank three group and $Span_{\mathbb{Z}}\{x, rx, sx\} \subset \Pi(G)$. Let $d, e, f, g, h, j \in \mathbb{C}$ be such that:

$$g_1 = \begin{bmatrix} 1 & d & e \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}, \ g_2 = \begin{bmatrix} 1 & f & g \\ 0 & 1 & rx \\ 0 & 0 & 1 \end{bmatrix}, \ g_3 = \begin{bmatrix} 1 & u & v \\ 0 & 1 & sx \\ 0 & 0 & 1 \end{bmatrix} \in G.$$

A straightforward computation shows that for every $k, l, m, n, o, p \in \mathbb{Z}$:

$$g_1^k g_2^l g_3^{m-p} g_2^{-o} g_1^{-n} g_3^{p-m} g_2^{o-l} g_1^{n-k} = \begin{bmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $w = x(dn((l-o)r + s(m-p)) - u(m-p)(or+n) + f(os(m-p) + n(-l+o)))$. Thus

$$h_1 = \begin{pmatrix} 1 & 0 & x(ds-h) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ h_2 = \begin{pmatrix} 1 & 0 & x(dr-f) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$h_3 = \begin{pmatrix} 1 & 0 & x(fs-ur) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G.$$

Since $fs - ur = -s(dr - f) + r(ds - u)$, we conclude that $Span_{\mathbb{Z}}\{ds - u, dr - f, fs - ur\}$ is non-discrete. Thus G is non-discrete, which is a contradiction.

Now we prove Part (2). From Lemma 7.5 and Proposition 5.25 we get that $rank(Ker(\Pi|_G)) = 2$; on the other hand, by Theorem 0.3 we have $3 \leq rank(\Pi(G)) \leq 4$. $\square$

Proposition 5.25. *Let* $G \subset \mathrm{Heis}(3, \mathbb{C})$ *be a non-Abelian discrete group. Then* $\Pi(G)$ *has rank 3 and is dense in* $\mathbb{C}$ *if and only if we can find a rank two additive discrete subgroup* $W \subset \mathbb{C}$, *and* $x, a, b, c, d, e, f \in \mathbb{C}$ *such that* G *is conjugate to:*

$$H = \left\{ \begin{bmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & x \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^k \begin{bmatrix} 1 & a+c & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}^l \begin{bmatrix} 1 & d+f & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix}^m : k, m, n \in \mathbb{Z}, w \in W \right\},$$

where a, b, c, d, e, f are subject to the conditions:

(1) $\{a, d, af - dc\} \subset W$;
(2) $|a| + |d| \neq 0$;
(3) for every real line $\ell \subset \mathbb{C}$ we have $\ell \cap Span_{\mathbb{Z}}\{1, c, f\}$ has rank at most two.

Proof. For $w \in W$ and $k, l, m \in \mathbb{Z}$, define:

$$g(w, k, l, m) = \begin{bmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & x \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^k \begin{bmatrix} 1 & a+c & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}^l \begin{bmatrix} 1 & d+f & e \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix}^m .$$

Let $u, v \in W$ and $k, l, m, o, p, q \in \mathbb{Z}$; a straightforward computation shows:

$$g(u, k, l, m)g(v, o, p, q)^{-1} = g(u - v + w, k - o, l - p, m - q),$$

where $w = -aln + ao(n + f(m - p)) - d(n + co)(m - p)$. Thus H is a group if and only if $a, d, af - dc \in W$. We notice that if H is a group, then H is non commutative if and only if $|a| + |d| \neq 0$. Clearly $\Pi(H)$ has rank three and it is a dense group in $\mathbb{C}$ whenever $Span_{\mathbb{Z}}(\{1, c, d\})$ is dense in $\mathbb{C}$. Observe that if H is non-commutative and $Span_{\mathbb{Z}}\{1, c, f\}$ is dense in $\mathbb{C}$, then H is discrete if and only if there are sequences $(w_n) \subset W$ and $(k_n), (l_n), (m_n) \subset \mathbb{Z}$ such that $(g_n = g(w_n, k_n, l_n, m_n))$ is a sequence of distinct elements satisfying $g_n \xrightarrow[n \to \infty]{} Id$. By Lemma 7.5, H is non-discrete if and only if W is non-discrete or there are sequences $(w_n) \subset W$ and $(k_n), (l_n), (m_n) \subset \mathbb{Z}$ such that:

(1) $(k_n + l_n c + m_n f)$ is a sequence of distinct elements converging to 0;
(2) $(l_n a + m_n d)$ converges to 0.

Now observe that Lemma 7.6 and the previous facts are equivalent to the non-discreteness of $Span_{\mathbb{Z}}(a, d)$. $\square$

Similar arguments show:

Proposition 5.26. Let $G \subset \text{Heis}(3, \mathbb{C})$ be a non-Abelian discrete group such that $\Pi(G)$ has rank four. Then there exist a rank two additive discrete subgroup $W \subset \mathbb{C}$ and $x, a, b, c, d, e, f, g, j \in \mathbb{C}$ such that G is conjugate to:

$$H = \left\{ g_u g_1^k g_2^l g_3^m g_4. : k, l, m, n \in \mathbb{Z}, w \in W \right\},$$

where

$$g_u = \begin{bmatrix} 1 & 0 & u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \ g_1 = \begin{bmatrix} 1 & 1 & x \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} ; \ g_2 = \begin{bmatrix} 1 & a+c & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} ;$$

$$g_3 = \begin{bmatrix} 1 & d+f & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} ; \ g_4 = \begin{bmatrix} 1 & g+j & h \\ 0 & 1 & j \\ 0 & 0 & 1 \end{bmatrix} ,$$

and $x, a, b, c, d, e, f, g, h, j$ are subject to the conditions:

(1) $\{a, d, g, dj - gf, af - cd, aj - cg\} \subset W$;
(2) $|a| + |d| + |g| \neq 0$;
(3) *for every real line $\ell \subset \mathbb{C}$ we have $\ell \cap Span_{\mathbb{Z}}(\{1, c, f, j\})$ has rank at most two.*

5.5. Proof of Theorem 3.2

Let G be a discrete group without loxodromic elements which is not (complex) Kleinian. By Corollary 4.4 and Lemma 4.10, G contains a torsion free subgroup G_0 of finite index which is triangularizable, it is not Kleinian and the following groups are torsion free: $G_0, \lambda_{12}(G_0), \lambda_{23}(G_0), \Pi(G_0), Ker(\Pi|_{G_0})$. Theorem 4.5 implies that G_0 does not contain an irrational ellipto-parabolic element, for otherwise the group G_0 would be Kleinian. So we can assume that $G_0 \subset \text{Heis}\,(3, \mathbb{C})$.

If G_0 is commutative, then Theorem 4.16 and Lemma 5.4 imply that $\Pi(G_0)$ is trivial, and the result follows from Lemma 5.11.

If G_0 is non-commutative, then $\Pi(G_0)$ may or may not be discrete. If $\Pi(G_0)$ is discrete, then the result follows by Proposition 5.21. If $\Pi(G_0)$ is non-discrete, then by Lemma 5.24 we have that $\Pi(G_0)$ has rank 2, 3 or 4. Let us look at each of these cases:

If the group $\Pi(G)$ has rank two, then the result follows from Proposition 5.23.

If the group $\Pi(G)$ has rank three, then the result follows from Lemma 7.7 and Proposition 5.25.

If the group $\Pi(G)$ has rank four, then the result follows from Proposition 5.26 and Lemma 7.9. □

Proof of Theorem 0.1. This uses Theorems 3.1, 3.2, 3,2 and section 2 where the families of purely parabolic groups are described.

Proof of part (1). If G is a complex Kleinian group then by Theorem 3.1, G is virtually conjugate to either an elliptic group or to a discrete subgroup of Heis(3, $\mathbb{C}$). If G is discrete but non-Kleinian, then by Theorem 3.2 we get that G is virtually conjugate to a discrete subgroup of Heis(3, $\mathbb{C}$).

Proof of part (2) items (a) and (b). If G is complex Kleinian then, by Theorem 3.1, G is virtually conjugate to one and only one of the following groups: Elliptic, Torus, dual torus group type I and type II, Kleinian Inoue, $\mathcal{K}_0, \mathcal{K}_1$ and $\mathcal{K}_2$. By Lemmas 2.1, 2.2, 2.3, 2.5, 2.13, 2.16, 2.17, 2.18 and Definition 2.4 the only groups whose limit set is exactly one line are: Elliptic groups, Torus groups, dual Torus type I groups, $\mathcal{K}_0$ groups and $\mathcal{K}_1$ groups.

And by Lemmas 2.3, 2.16, 2.17, 2.18, Theorem 2.7 and Definition 2.4 those groups whose limit set is a cone of lines over a circle are: dual torus type II groups, Kleinian Inoue groups, and $\mathcal{K}_2$ groups.

The proof of Part (2) Item (C) follows directly from Theorem 3.2 and the corresponding list is the following dual torus group type III, non Kleinian Inoue groups, Extended Inoue groups and $\mathcal{K}_3$, $\mathcal{K}_4$ and $\mathcal{K}_5$ groups. $\square$

6. Appendix: Abelian subgroups of U^+

Since we have not found Theorem 4.16 in the literature, we now prove it for completeness. The claim is that if U_+ is the group in Definition 4.9 and $G \subset U_+$ is a commutative subgroup, then G is conjugate to a subgroup $\widetilde{G}$ of one of the Abelian Lie groups C_j in Definition 4.15, for some $j = 1, 2, 3, 4, 5$. Notice that since G is commutative, we have that $\Pi^*(G)$ and $\Pi(G)$ are Abelian. Now consider the following cases:

Case 1. The groups $\Pi^*(G)$ and $\Pi(G)$ contain a parabolic element. In this case, since $\Pi^*(G), \Pi(G) \subset Mob(\mathbb{C})$ are Abelian, we deduce that $\Pi^*(G)$ and $\Pi(G)$ are purely parabolic, *i.e.*, $G \subset Ker(\lambda_{12}) \cap Ker(\lambda_{13})$.

Claim 1. There is an element $h \in G$ such that $\Pi(h)$ and $\Pi^*(h)$ are parabolic. Let $g_1, g_2 \in G$ be such that $\Pi(g_1)$ and $\Pi(g_2)$ are parabolic, then, taking a power of g_2 if necessary, we can assume that $\Pi(g_1 g_2)$ and $\Pi^*(g_1 g_2)$ are not the identity. Since $G \subset Ker(\lambda_{12}) \cap Ker(\lambda_{13})$ we deduce that $\Pi(g_1 g_2)$ and $\Pi^*(g_1 g_2)$ are both parabolic.

Let $h \in G$ be the element given by the previous claim, then

$$h = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

where $ac \neq 0$. Let us define $h_0 \in \mathrm{PSL}(3, \mathbb{C})$ by

$$h_0 = \begin{bmatrix} a^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix}.$$

Then a straightforward computation shows that for every $g = [g_{ij}] \in h_0 G h_0^{-1}$ we have:

$$[h_0 h h_0^{-1}, g] = \begin{bmatrix} 1 & 0 & -g_{12} + g_{23} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since G is Abelian we deduce $g_{12} = g_{23}$.

Case 2. The group $\Pi^*(G)$ contains a parabolic element but $\Pi(G)$ does not. Under this assumption, we deduce $\Pi^*(G)$ is purely parabolic and there exists $w \in \mathbb{C}$ such that $\Pi(G)w = w$, hence $G \subset Ker(\lambda_{12})$. We define

$$h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & w \\ 0 & 0 & 1 \end{bmatrix}.$$

By a straightforward computation we show that for every $g \in G$, there exists $c_g \in \mathbb{C}$ such that:

$$hgh^{-1} = \begin{bmatrix} g_{11} & g_{12} & c_g \\ 0 & g_{11} & 0 \\ 0 & 0 & g_{11}^{-2} \end{bmatrix}.$$

We notice that $G_1 = hGh^{-1}$ leaves invariant the line $\overleftrightarrow{e_1, e_3}$, so $\Pi_1 : G_1 \to Mob(\mathbb{C})$, given by $\Pi_1([g_{ij}]) = g_{11}g_{33}^{-1}z + g_{13}g_{33}^{-1}$, is a well defined group morphism. Now we only need to consider the following sub-cases:

Sub case 1. The group $\Pi_1(G_1)$ contains a parabolic element. Then $\Pi_1(G_1)$ is purely parabolic, which shows that $G_1 \subset Ker(\lambda_{13})$.

Sub case 2. The group $\Pi_1(G_1)$ does not contain a parabolic element. Then there exists $p \in \mathbb{C}$ such that $\Pi_1(G_1)p = p$. We define

$$h_1 = \begin{bmatrix} 1 & 0 & p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is clear that for every $g \in G_1$ we have

$$h_1hgh^{-1}h_1^{-1} = \begin{bmatrix} g_{11} & g_{12} & 0 \\ 0 & g_{11} & 0 \\ 0 & 0 & g_{11}^{-2} \end{bmatrix}.$$

It follows that in this case the group is conjugate to a subgroup in C_1.

Case 3. The group $\Pi^*(G)$ does not contain a parabolic element but $\Pi(G)$ does. We deduce that $\Pi(G)$ is purely parabolic and there exists $z \in \mathbb{C}$ such that $\Pi^*(G)z = z$. Clearly $G \subset Ker(\lambda_{23})$; we define

$$h = \begin{bmatrix} 1 & z & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then for every $g \in G$ there exists c_g such that:

$$hgh^{-1} = \begin{bmatrix} g_{11}^{-2} & 0 & c_g \\ 0 & g_{11} & g_{13} \\ 0 & 0 & g_{11} \end{bmatrix}.$$

Now we can consider $\Pi_2 = \Pi_{e_2, \overleftrightarrow{e_1, e_3}}$ and we have that $\Pi_2(G) \subset Mob(\mathbb{C})$ is an Abelian group. So we must consider the following sub cases:

Sub-case 1. The group $\Pi_2(G)$ contains a parabolic element. We get that $\Pi_2(G)$ is purely parabolic, which shows that $G \subset Ker(\lambda_{13})$.

Sub-case 2. The group $\Pi_2(G)$ does not contain a parabolic element. Again there exists $p \in \mathbb{C}$ such that $\Pi_2(G)p = p$. Define

$$
h_1 = \begin{bmatrix} 1 & 0 & p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
$$

One can show that for every $g \in G$:

$$
h_1 h g h^{-1} h_1^{-1} = \begin{bmatrix} g_{11}^{-2} & 0 & 0 \\ 0 & g_{11} & g_{13} \\ 0 & 0 & g_{11} \end{bmatrix}.
$$

Case 4. The groups $\Pi^*(G)$ and $\Pi(G)$ do not contain parabolic elements. In this setting there are $z, w \in \mathbb{C}$ such that $\Pi^*(G)z = z$ and $\Pi(G)w = w$. Define

$$
h = \begin{bmatrix} 1 & z & 0 \\ 0 & 1 & w \\ 0 & 0 & 1 \end{bmatrix}.
$$

Then for every $g = [g_{ij}] \in G$ there exists $c_g \in \mathbb{C}$ such that:

$$
h g h^{-1} = \begin{bmatrix} g_{11} & 0 & c_g \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{bmatrix}.
$$

Consider the following sub-cases:

Sub case 1. The group $\Pi_2(G)$ contains a parabolic element. Then $\Pi_2(G)$ is purely parabolic, which shows that $G \subset Ker(\lambda_{13})$.

Sub case 2. The group $\Pi_2(G)$ does not contain a parabolic element. We know there exists $p \in \mathbb{C}$ such that $\Pi_2(G)p = p$, let

$$
h_1 = \begin{bmatrix} 1 & 0 & p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
$$

Then the subgroup $h_1 h G h^{-1} h_1^{-1}$ contains only diagonal elements. $\quad\square$

7. Appendix: Technical lemmas on additive subgroups of $\mathbb{C}^2$

Now we state and prove some technical lemmas used along this work.

Lemma 7.1. *Let $\vartheta \in \mathbb{S}^1 \setminus \{\pm 1\}$, then*

$$\beta_1 = \{(0,1), (\vartheta, \vartheta), (2\vartheta^2, \vartheta^2), (3\vartheta^3, \vartheta^3)\} \subset \mathbb{C}^2,$$

is an $\mathbb{R}$-linearly independent set.

Proof. If $\vartheta = \cos(\theta) + i\sin(\theta)$, then β_1 is $\mathbb{R}$-linearly independent because the determinant

$$\begin{vmatrix} 0 & 0 & 1 & 0 \\ \cos(\theta) & \sin(\theta) & \cos(\theta) & \sin(\theta) \\ 2\cos(2\theta) & 2\sin(2\theta) & \cos(2\theta) & \sin(2\theta) \\ 3\cos(3\theta) & 3\sin(3\theta) & \cos(3\theta) & \sin(3\theta) \end{vmatrix} = 4(\sin\theta)^4$$

is equal to 0 if and only if $\vartheta = \pm 1$. $\square$

Lemma 7.2. *Let $\vartheta \in \mathbb{S}^1 \setminus \{\pm 1\}$ be a complex number satisfying:*

(1) *The set*

$$\beta_2 = \{(0,1), \vartheta(1,1), \vartheta^2(2,1), \vartheta^3(3,1), \vartheta^4(4,1)\} \subset \mathbb{C}^2,$$

   is $\mathbb{Q}$-linearly dependent;
(2) *The number $Re(\vartheta)$ is not a root of the polynomial:*

$$192x^7 - 64x^6 + 496x^5 + 288x^4 + 510x^3 + 209x + 8.$$

Then there exists $\alpha \in \mathbb{C}^*$ such that:

$$(\alpha, 0) \in Span_{\mathbb{Z}}\{\vartheta^j(j,1) : j \in \{0, \ldots, 5\}\}.$$

Proof. Since β_2 is a $\mathbb{Q}$-linearly dependent set, there are $m_0, m_1, m_2, m_3, m_4 \in \mathbb{Z}$ such that:

$$m_4\vartheta^4(4,1) = m_3\vartheta^3(3,1) + m_2\vartheta^2(2,1) + m_1\vartheta(1,1) + m_0(0,1),$$

and $m_4 \neq 0$. Thus we get the following equations:

$$4m_4\vartheta^3 = 3m_3\vartheta^2 + 2m_2\vartheta + m_1,$$
$$m_4\vartheta^4 = m_3\vartheta^3 + m_2\vartheta^2 + m_1\vartheta + m_0.$$

Let us consider $p_1(x), p_2(x) \in \mathbb{Z}[x]$ given by:

$$p_1(x) = -4m_4x^3 + 3m_3x^2 + 2m_2x + m_1,$$
$$p_2(x) = -m_4x^4 + m_3x^3 + m_2x^2 + m_1x + m_0.$$

Clearly $p_1(\vartheta) = p_2(\vartheta) = 0$, so there are $r_1, r_2, r_3 \in \mathbb{R}$ such that:

$$
\begin{aligned}
p_1(x) &= -4m_4(x - \vartheta)(x - \vartheta^{-1})(x - r_1) \\
&= -4m_4 x^3 + 4m_4(2Re(\vartheta) + r_1)x^2 - 4m_4(1 + 2r_1 Re(\vartheta))x + 4m_4 r_1\,, \\
p_2(x) &= -4m_4(x - \vartheta)(x - \vartheta^{-1})(x^2 + r_2 x + r_3) \\
&= -4m_4 x^4 - 4m_4(-2Re(\vartheta) + r_2)x^3 - 4m_4(1 - 2r_2 Re(\vartheta) + r_3)x^2 \\
&\quad -4m_4(r_2 - 2Re(\vartheta)r_3)x - 4m_4 r_3\,.
\end{aligned}
$$

By comparing the coefficients of p_1 and p_2 with the previous equations we get:

$$
\begin{aligned}
m_1 &= 4m_4 r_1\,, \\
2m_2 &= -4m_4(1 + 2r_1 Re(\vartheta))\,, \\
3m_3 &= 4m_4(2Re(\vartheta) + r_1)\,, \\
m_0 &= -4m_4 r_3\,, \\
m_1 &= -4m_4(r_2 - 2r_3 Re(\vartheta))\,, \\
m_2 &= -4m_4(1 - 2r_2 Re(\vartheta) + r_3)\,, \\
m_3 &= -4m_4(r_2 - 2Re(\vartheta))\,.
\end{aligned}
\tag{7.1}
$$

This yields the following linear system:

$$
\begin{aligned}
r_1 + r_2 - 2r_3 Re(\vartheta) &= 0\,, \\
2r_1 Re(\vartheta) + 4r_2 Re(\vartheta) - 2r_3 &= 1\,, \\
r_1 + 3r_3 &= 4Re(\vartheta)\,.
\end{aligned}
$$

Solving the system by Cramer's rule we get:

$$
r_1 = \frac{Re(\vartheta)(16Re^2(\vartheta) - 7)}{2(1 - Re^2(\vartheta))}\,; \quad
r_2 = \frac{-Re(\vartheta)(8Re^2(\vartheta) + 3)}{2(1 - Re^2(\vartheta))}\,; \quad
r_3 = \frac{4Re^2(\vartheta) - 1}{2(1 - Re^2(\vartheta))}\,.
\tag{7.2}
$$

On the other hand, from the first 3 equations in the System (7.1) we deduce that $Re(\vartheta) = pq^{-1}$, where $p, q \in \mathbb{Z}$ are co-primes. Let us define:

$$
\begin{aligned}
n_0 &= (4Re^2(\vartheta) - 1)q^4\,, \\
n_1 &= (-Re(\vartheta) - 16Re^3(\vartheta)))q^4\,, \\
n_2 &= (1 + 8Re^2(\vartheta) + 16Re^4(\vartheta))q^4\,, \\
n_3 &= (-7Re(\vartheta) - 4Re^3(\vartheta))q^4\,, \\
n_4 &= (2 - 2Re^2(\vartheta))q^4\,,
\end{aligned}
$$

which are integers. From Equation (7.2) we deduce:

$$
n_4 \vartheta^4(4, 1) = n_3 \vartheta^3(3, 1) + n_2 \vartheta^2(2, 1) + n_1 \vartheta(1, 1) + n_0(0, 1)\,.
$$

This implies:

$$\vartheta^5 = n_4^{-2}(n_0 n_3 + (n_0 n_4 + n_1 n_3)\vartheta + (n_1 n_4 + n_2 n_3)\vartheta^2 + (n_2 n_4 + n_3^2)\vartheta^3) .$$

Thus:

$$
\begin{aligned}
n_4^2 \vartheta^5 (5,1) &= n_0 n_3 (0,1) + (n_0 n_4 + n_1 n_3)\vartheta(1,1) + (n_1 n_4 + n_2 n_3)\vartheta^2(2,1) \\
&\quad + (n_2 n_4 + n_3^2)\vartheta^3(3,1) + (5 n_0 n_3 + 4(n_0 n_4 + n_1 n_3)\vartheta + 3(n_1 n_4 + n_2 n_3)\vartheta^2 \\
&\quad + 2(n_2 n_4 + n_3^2)\vartheta^3)(1,0) .
\end{aligned}
$$

Finally let us show that

$$5 n_0 n_3 + 4(n_0 n_4 + n_1 n_3)\vartheta + 3(n_1 n_4 + n_2 n_3)\vartheta^2 + 2(n_2 n_4 + n_3^2)\vartheta^3 \neq 0 .$$

We define $p_3(x) = 5 n_0 n_3 + 4(n_0 n_4 + n_1 n_3)x + 3(n_1 n_4 + n_2 n_3)x^2 + 2(n_2 n_4 + n_3^2)x^3$. We need to show $p_3(\vartheta) \neq 0$. Assume, on the contrary, that $p_3(\vartheta) = 0$.

Now we notice:

$$n_2 n_4 + n_3^2 = q^8 (2 + 63 Re^2(\vartheta) + 72 Re^4(\vartheta) - 16 Re^6(\vartheta)) \neq 0.$$

Hence $p_3(x)$ is a cubic polynomial with coefficients in $\mathbb{Z}$. Finally, since ϑ is a root of $p_3(x)$, there exists $r_0 \in \mathbb{R}$ such that

$$
\begin{aligned}
p_3(x) &= 2(n_2 n_4 + n_3^2)(x - \vartheta)(x - \vartheta^{-1})(x - r_0) \\
&= 2(n_2 n_4 + n_3^2)(x^3 - (r_0 + 2 Re(\vartheta))x^2 + (1 + 2 r_0 Re(\vartheta))x - r_0) .
\end{aligned}
$$

By comparing the quadratic coefficients of p_3 we obtain:

$$-2(n_2 n_4 + n_3^2)(2 Re(\vartheta) + r_0) = 3(n_1 n_4 + n_2 n_3) .$$

Substituting the values of the n_i's we get the following equivalent equation:

$$192 Re(\vartheta)^7 - 64 Re(\vartheta)^6 + 496 Re(\vartheta)^5 + 288 Re(\vartheta)^4 + 510 Re(\vartheta)^3 + 209 Re(\vartheta) + 8 = 0 ,$$

which contradicts our initial hypothesis. $\quad\square$

Lemma 7.3. *Let $\mathcal{L} \subset \mathbb{C}^2$ be an additive subgroup such that for each $x, y \in \mathcal{L}$ we have $\pi_1(x)\pi_2(y) = \pi_1(y)\pi_2(x)$, then:*

(1) *If $Ker(\pi_1) \cap \mathcal{L}$ and $Ker(\pi_2) \cap \mathcal{L}$ are trivial, then there is $\mu \in \mathbb{C}^*$ and W an additive group of $\mathbb{C}$ such that $\mathcal{L} = \{r(1, \mu) : r \in W\}$.*
(2) *If $Ker(\pi_1) \cap \mathcal{L}$ in non trivial, then there is an additive group W of $\mathbb{C}$ such that $\mathcal{L} = \{(r, 0) : r \in W\}$.*
(3) *If $Ker(\pi_2) \cap \mathcal{L}$ is non-trivial, then there is an additive group W of $\mathbb{C}$ such that $\mathcal{L} = \{(0, r) : r \in W\}$.*

Proof. Let us show (1). Clearly $\mathcal{L} = \{(\pi_1(x), \pi_2(x)) : x \in \mathcal{L}\}$. Let us define $\mu_x = \pi_2(x)/\pi_1(x)$; by hypothesis μ_x does not depend on x, then

$$\mathcal{L} = \{(\pi_1(x), \pi_1(x)\mu_x) : x \in \mathcal{L}\}.$$

In order to prove (2) it is enough to show that $\pi_2(\mathcal{L})$ is trivial. Assume on the contrary that there exists $y \in \mathcal{L}$ such that $\pi_2(y) \neq 0$. Consider an element $x \in Ker(\pi_2) \cap \mathcal{L} - \{\mathbf{0}\}$, thus $0 \neq \pi_1(x)\pi_2(y) = \pi_1(y)\pi_2(x) = 0$, which is a contradiction. The proof of Part (3) is similar. $\quad \square$

Lemma 7.4. *Let $\mathcal{L} = \{(1,0),(c,d)\} \subset \mathbb{C}^2$ be an $\mathbb{R}$-linearly independent set. Then $(0,1),(0,c) \in Span_{\mathbb{Z}}(\mathcal{L})$ if and only if there are $p, q, r \in \mathbb{N}$ such that p, q are co-primes, q^2 divides r, $c = pq^{-1}$, and $d = r^{-1}$.*

Proof. Since $(0,1),(0,c) \in Span_{\mathbb{Z}}(\mathcal{L})$ we deduce that there are $k_1, k_2, k_3, k_4 \in \mathbb{Z}$ such that

$$k_1 + k_2 c = 0,$$
$$k_2 d = 1,$$
$$k_3 + k_4 c = 0,$$
$$k_4 d = c.$$

From the first two equations we deduce $d = k_2^{-1}$, $c = -k_1 k_2^{-1}$. Let $p, q \in \mathbb{N}$ be co-primes such that $c = pq^{-1}$; substituting in the last two equations we get:

$$k_3 q + k_4 p = 0,$$
$$k_4 q = p k_2.$$

From the first equation we see that q divides k_4, thus there exists $m \in \mathbb{Z}$ such that $k_4 = qm$; substituting in the last equation we get:

$$mq^2 = pk_2.$$

It follows that q^2 divides k_2. Conversely, let us assume that p, q are co-primes such that $c = pq^{-1}$ and $r = q^2 n$, then:

$$-p^2 n(1,0) + qpn(pq^{-1}, (q^2 n)^{-1}) = (0, pq^{-1}),$$
$$-pqn(1,0) + qqn(pq^{-1}, (q^2 n)^{-1}) = (0,1). \quad \square$$

Lemma 7.5. *Let $a, c, d, f \in \mathbb{C}$ be such that $|a| + |d| \neq 0$, $Span_{\mathbb{Z}}\{1, c, f\}$ is a rank three group and for every real subspace $\ell \subset \mathbb{C}$ we have that $Span_{\mathbb{Z}}\{1, c, f\} \cap \ell$ has rank at most two. Then $Rank(Span_{\mathbb{Z}}\{a, d, af - dc\}) \geq 2$.*

Proof. The result is trivial if $a = 0$ or $d = 0$ or $ad^{-1} \notin \mathbb{Q}$, so we assume that $0 \neq a = rd$ for some $r \in \mathbb{Q}$. We consider the following cases:

Case 1. $Span_{\mathbb{Z}}\{1, c, f\}$ is dense in $\mathbb{C}$. Then we must have $f = s + tc$ where $s, t \in \mathbb{R}$ and $\{1, s, t\}$ is $\mathbb{Q}$-linearly independent. Thus, $af - dc = (rs + (rt - 1)c)d$; to conclude we observe that $rs + (rt - 1)c \notin \mathbb{R}$.

Case 2. $c \in \mathbb{R} - \mathbb{Q}$ and $f \notin \mathbb{R}$. Then we have $af - dc = (rf - c)d$; to finish we observe that $rf - c \notin \mathbb{R}$. $\square$

Lemma 7.6. *Let $a, c, d, f \in \mathbb{C}$ be such that $|a| + |d| \neq 0$, $Span_{\mathbb{Z}}\{1, c, f\}$ is dense in $\mathbb{C}$ and there are sequences $(k_n), (l_n), (m_n) \subset \mathbb{Z}$ such that:*

(1) $(k_n + l_n c + m_n f)$ is a sequence of distinct elements converging to 0;
(2) $(l_n a + m_n d)$ converges to 0.

Then $Span_{\mathbb{Z}}\{a, d\}$ is non-discrete.

Proof. Assume, on the contrary, that $Span_{\mathbb{Z}}\{a, d\}$ is discrete. Then $ad \neq 0$ and $qa = pd$ where p, q are non-zero integers. On the other hand, since $Span_{\mathbb{Z}}\{a, d\}$ is discrete, by Assumption (2) we have $l_n a + m_n d = 0$ for n large, so we can assume that $l_n p + m_n q = 0$ for n large. Hence:

$$\frac{k_n}{m_n} \xrightarrow[n \to \infty]{} \frac{-fp + cq}{p} \quad .$$

That is $-fp + cq \in \mathbb{R}$. On the other hand, since $f = r + sc$ where $r, s \in \mathbb{R}$ satisfies that $\{1, r, s\}$ is $\mathbb{Q}$-linearly independent, we deduce $-fp + cq = -(r + sc)p + cq \in \mathbb{R}$. Thus $c \in \mathbb{R}$, which is a contradiction. $\square$

In the next lemma we consider a condition as in Example 2.5.3:

Lemma 7.7. *Let $a, c, d, f \in \mathbb{C}$ be such that $a \neq 0$ and for every real line $\ell \subset \mathbb{C}$ we have that $\ell \cap Span_{\mathbb{Z}}\{1, c, f\}$ has rank at most two. Then $U = Span_{\mathbb{Z}}\{a, d, af - dc\}$ is discrete if and only if $Span_{\mathbb{Z}}\{a, d\}$ is discrete and one of the following statements is true:*

(1) $d = 0$ and $f \notin \mathbb{R}$;
(2) $a = rd$ for some $r \in \mathbb{Q}$;
(3) $ad^{-1} \notin \mathbb{R}$, and there are $r_1, r_2 \in \mathbb{Q}$ such that

$$c = \frac{a(f - r_1)}{d} - r_2.$$

Proof. It is clear that if $d = 0$ and $f \notin \mathbb{R}$, then U is discrete. So we assume that if $a = rd$ with $r \in \mathbb{Q}$, then $U = dSpan_{\mathbb{Z}}\{r, 1, rf - c\}$. Since $f = s_1 + s_2 c$ where $s_1, s_2 \in \mathbb{R}$ satisfy

that $\{1, s_1, s_2\}$ is $\mathbb{Q}$-linearly independent, then $rf - c = rs_1 + (s_2 - 1)c \in \mathbb{R}$ if and only if $s_2 = 1$, which is not possible. Thus U is discrete. Finally observe that if $ad^{-1} \notin \mathbb{R}$, then the equation

$$\frac{a}{d} = \frac{c + r_2}{f - r_1}$$

is equivalent to the discreteness of U. □

Remark 7.8. We need that the closure of every rank 3 subgroup in $W = Span_{\mathbb{Z}}\{1, c, f\}$ be either dense in $\mathbb{C}$ or isomorphic as a Lie group to $\mathbb{R} \oplus \mathbb{Z}$. This comes from the fact that W is going to play the role of a control group, and we know that control groups of rank 3 satisfy this. In particular, there are no control groups of rank 3 which are dense subgroups and isomorphic to $\mathbb{R}$ (cf. [27]).

Lemma 7.9. *Let $a, c, d, g, f, h \in \mathbb{C}$ be such that:*

(1) $|a| + |d| + |g| \neq 0$.
(2) $W = Span_{\mathbb{Z}}(\{1, c, f, h\})$ *is a rank four group.*
(3) *For every 1-dimensional real subspace $\ell \subset \mathbb{C}$ we have $rank(\ell \cap W) \leq 2$.*
(4) $\overline{Span_{\mathbb{Z}}\{1, c, f\}} = \alpha\mathbb{R} \oplus \beta\mathbb{Z}$ *where $\alpha, \beta \in \mathbb{C}^*$ and $\alpha\beta^{-1} \notin \mathbb{R}$.*

Let $U = Span_{\mathbb{Z}}\{a, d, g, dh - gf, af - cd, ah - cg\}$, then U is discrete if and only if $Span_{\mathbb{Z}}\{a, d, g\}$ is discrete, $(|a| + |d|)(|a| + |c|)(|c| + |d|) \neq 0$ and one of the following occurs:

(1) $a = 0$ *(resp. $d = 0$, $g = 0$) and there are $r_0, r_1, r_2, r_3 \in \mathbb{Q}$ such that $r_1 \neq 0$ and*

$$(r_2 - r_0)^2 + 4r_1r_3 < 0;$$

$$x_1 = \frac{r_2 + r_0 \pm \sqrt{(r_2 - r_0)^2 + 4r_1r_3}}{2};$$

$$x_2 = x_3 \left(\frac{r_2 - r_0 \pm \sqrt{(r_2 - r_0)^2 + 4r_1r_3}}{2r_1} \right);$$

$$x_4 = (x_5 - r_4) \left(\frac{r_2 - r_0 \pm \sqrt{(r_2 - r_0)^2 + 4r_1r_3}}{2r_1} \right) - r_5$$

   where $x_1 = c$ (resp. f, h), $x_2 = d$ (resp. a, a), $x_3 = g$ (resp. g, d), $x_4 = f$ (resp. c, c), and $x_5 = h$ (resp. h, f).
(2) $ad^{-1} \notin \mathbb{R}$ *and there are $r_1, r_2, s_1, t_1, s_2, t_2, s_3, t_3 \in \mathbb{R}$ such that:*

$$g = r_1a + r_2d \quad ; \quad r_2t_2 \neq t_3,$$

$$f = \frac{A_2 \pm (c + t_2)\sqrt{A_1}}{2(r_2 t_2 - t_3)} \quad ; \quad j = \frac{A_3 \pm (c r_2 + t_3)\sqrt{A_1}}{2(r_2 t_2 - t_3)},$$

where:

$$A_1 = (-r_2 s_2 + r_1 t_2 - s_3 - t_1)^2 - 4(r_2 s_1 t_2 - r_1 s_2 t_3 + r_2 s_2 s_3 - r_1 t_1 t_2 + s_3 t_1 - s_1 t_3),$$

$$A_2 = -c r_2 s_2 - c r_1 t_2 + c s_3 - c t_1 + r_2 s_2 t_2 - r_1 t_2^2 + s_3 t_2 - 2 s_2 t_3 - t_1 t_2,$$

$$A_3 = r_2 (c r_1 t_2 + s_3 (c + 2t_2) - c t_1 - s_2 t_3) + t_3 (-r_1 (2c + t_2) - s_3 - t_1) - c r_2^2 s_2.$$

(3) $ad^{-1} \notin \mathbb{Q}$ and $gd^{-1} \notin \mathbb{Q}$ and there are $r_2, s_1, s_2, s_3, t_1, t_2, t_3 \in \mathbb{Q}$ such that $r_2 t_2 \neq 0$, $a = r_2 d$; and

$$c = \frac{1}{2}\left(A_3 \mp \sqrt{A_1}\right) \quad ; \quad j = \frac{A_2 \pm \sqrt{A_1}(f + t_1)}{2t_2},$$

where

$$A_1 = 2 r_2 s_2 t_1 - 4 r_2 s_1 t_2 + r_2^2 t_1^2 - 2 r_2 t_1 t_3 - 2 s_2 t_3 + 4 s_3 t_2 + s_2^2 + t_3^2,$$
$$A_2 = -f r_2 t_1 - f s_2 + f t_3 - r_2 t_1^2 - s_2 t_1 + 2 s_1 t_2 + t_3 t_1,$$
$$A_3 = 2 f r_2 + r_2 t_1 - s_2 - t_3.$$

Proof. Let us assume U is discrete. Clearly, $Span_{\mathbb{Z}}\{a, d, g\}$ is discrete; we claim:

Claim 1. $|d| + |g| \neq 0$: just notice that $g = d = 0$ implies that $U = aSpan_{\mathbb{Z}}\{1, f, h\}$ is not discrete. Similarly one has that $|d| + |a| \neq 0$ and $|a| + |g| \neq 0$.

Claim 2. There are not $r_0, r_1 \in \mathbb{Q}$ such that $a = r_0 d$, $g = r_1 d$. Assume on the contrary that there are such $r_0, r_1 \in \mathbb{Q}$. Set:

$$U = dSpan_{\mathbb{Z}}\{r_0, 1, r_1, h - r_1 f, r_0 f - c, r_0 h - c r_1\}.$$

Let us consider

$$U_2 = Span_{\mathbb{Z}}\{1, h - r_1 f, r_0 f - c, r_0 h - c r_1\}.$$

Observe that $h - r_1 f \notin \mathbb{R}$, for otherwise $h - r_1 f \in \mathbb{Q}$ and therefore $\{h, f\}$ are $\mathbb{Q}$-linearly dependent; since U is discrete we conclude that there are $r_1, r_2 \in \mathbb{Q}$ such that $r_0 f - c = r_1 + r_2(h - r_1 f)$, thus $\{1, c, f, h\}$ is $\mathbb{Q}$-linearly dependent, which is a contradiction.

From the previous claims we deduce $adg = 0$. Now let us study the case $a = 0$, the cases $d = 0$ or $g = 0$ are similar and we leave them for the reader.

Claim 3. It is not possible that $adg \neq 0$ and $ad^{-1} \notin \mathbb{R}$. Assume, on the contrary, that there are $r_1, r_2, s_1, s_2, s_3, t_1, t_2, t_3 \in \mathbb{Q}$ such that:

$$g = r_1 a + r_2 d,$$

$$dh - gf = s_1 a + t_1 d,$$

$$af - cd = s_2 a + t_2 d,$$
$$ah - cg = s_3 .$$

Substituting the value of g given in the first equation in the other equations we get:

$$\frac{a}{d} = \frac{h - fr_2 - t_1}{s_1 + fr_1} = \frac{t_2 + c}{f - s_2} = \frac{cr_2 + t_3}{h - cr_1 - s_3}.$$

Hence, we obtain the following system of polynomial equations:

$$(h - fr_2 - t_1)(f - s_2) = (s_1 + fr_1)(t_2 + c),$$
$$(h - fr_2 - t_1)(h - cr_1 - s_3) = (s_1 + fr_1)(cr_2 + t_3),$$
$$(t_2 + c)(h - cr_1 - s_3) = (f - s_2)(cr_2 + t_3).$$

A straightforward computation shows that this system has non-trivial solutions if and only if $r_2 t_2 \neq t_3$, and in that case the solutions are:

$$f_\pm = \frac{A_2 \pm (c + t_2)\sqrt{A_1}}{2(r_2 t_2 - t_3)} \quad , \quad h_\pm = \frac{A_3 \pm (cr_2 + t_3)\sqrt{A_1}}{2(r_2 t_2 - t_3)} ,$$

where:

$A_1 = (-r_2 s_2 + r_1 t_2 - s_3 - t_1)^2 - 4(r_2 s_1 t_2 - r_1 s_2 t_3 + r_2 s_2 s_3 - r_1 t_1 t_2 + s_3 t_1 - s_1 t_3),$
$A_2 = -cr_2 s_2 - cr_1 t_2 + cs_3 - ct_1 + r_2 s_2 t_2 - r_1 t_2^2 + s_3 t_2 - 2s_2 t_3 - t_1 t_2,$
$A_3 = r_2 (cr_1 t_2 + s_3 (c + 2t_2) - ct_1 - s_2 t_3) + t_3 (-r_1 (2c + t_2) - s_3 - t_1) - cr_2^2 s_2 ,$

and $\sqrt{A_1} \notin \mathbb{Q}$.

Similarly one finds that it is not possible to have $adg \neq 0$, $ad^{-1} \in \mathbb{Q}$ and $gd^{-1} \notin \mathbb{R}$.

Claim 4. If $a \neq 0$, then $w = dg^{-1} \notin \mathbb{R}$. Assume that there is $R \in \mathbb{Q} - \{0\}$ such that $d = Rg$. In this case $U = gSpan_{\mathbb{Z}}\{R, 1, Rh - f, cR, c\}$. Let us consider $U_2 = Span_{\mathbb{Z}}\{1, Rh - f, c\}$, since U is discrete we conclude that there are $R_1, R_2 \in \mathbb{Q}$ such that $Rh - f = R_1 + R_2 C$, thus $\{1, c, f, h\}$ is $\mathbb{Q}$-linearly dependent, which is a contradiction. Thus $W = \{d, g, dh - gf, cd, cg\}$. On the other hand there are $r_0, r_1, r_2, r_3 \in \mathbb{R}$ such that $c = r_0 + r_1 w = r_2 + r_3 w^{-1}$. Therefore

$$cg = r_0 g + dr_1,$$
$$cd = r_2 d + r_3 g .$$

Hence $r_0, r_1, r_2, r_3 \in \mathbb{Q}$. Since $r_0 + r_1 w = r_2 + r_3 w^{-1}$ we conclude that w is a solution of the polynomial $r_1 w^2 + (r_0 - r_2)w - r_3 = 0$, that is

$$w = \frac{r_2 - r_0 \pm \sqrt{(r_2 - r_0)^2 + 4r_1 r_3}}{2r_1}.$$

Finally, since U is discrete we deduce that there are $r_4, r_5 \in \mathbb{Q}$ such that $dh - gf = r_4 d + r_5 g$ which is equivalent to:

$$w = \frac{f + r_5}{h - r_4}. \quad \square$$

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