Complex Kleinian Groups

Generalizing the classical case

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Introduction

Kleinian groups are discrete subgroups of $\mathrm{PSL}(2, \mathbb{C})$, the group of biholomorphic automorphisms of the complex projective line $\mathbb{CP}^1 \cong \mathbb{S}^2$, acting properly and discontinuously on a non-empty region of $\mathbb{CP}^1$.

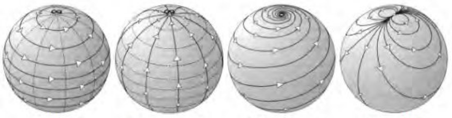
$$\begin{aligned} \mathrm{PSL}(2, \mathbb{C}) &= \left\{ \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] \middle| \, ad - bc \neq 0 \right\} \\ &\cong \left\{ z \mapsto \frac{az + b}{cz + d} \,\middle|\, ad - bc \neq 0 \right\} \end{aligned}$$

Importance of Kleinian Groups

Kleinian groups have been studied since the end of the 19th century by Fuchs, Klein, Poincaré, and many others. Kleinian groups have played a major role in several fields of mathematics, such as Riemann surfaces and Teichmüller theory, automorphic forms, holomorphic dynamics, conformal and hyperbolic geometry, etc.

Classification of Elements of PSL $(2, \mathbb{C})$

Type	Fixed points	Canonical Form	$Tr(A)$
Parabolic	1	$z \mapsto z + c$, $c \in \mathbb{C}$	$Tr(A) = \pm 2$
Elliptic	2	$z \mapsto \alpha z$, $\lvert\alpha\rvert = 1$	$Tr(A)^2 < 4$, $Tr(A) \in \mathbb{R}$
Hyperbolic	2	$z \mapsto \alpha z$, $\alpha > 0$	$Tr(A)^2 > 4$, $Tr(A) \in \mathbb{R}$
Loxodromic	2	$z \mapsto \alpha z$, $\lvert\alpha\rvert \neq 1$	$Tr(A) \notin \mathbb{R}$

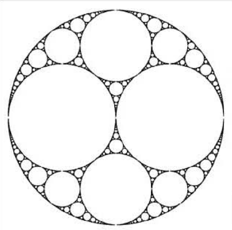


(a) elliptic (b) hyperbolic (c) loxodromic (d) parabolic

The Limit Set

Let $\Gamma \subset PSL(2, \mathbb{C})$ be a group and $z \in \mathbb{CP}^1$, then the orbit of z is $\{\gamma(z)\}_{\gamma \in \Gamma}$.

The set of accumulation points of orbits of a Kleinian group Γ is called the limit set of the group, $\Lambda(\Gamma)$.

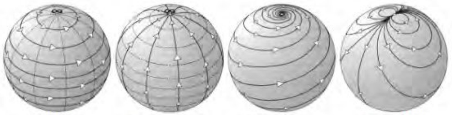


$\Lambda(\Gamma)$ is closed and invariant under Γ.

Elementary Kleinian Groups

Elementary Kleinian groups are discrete subgroups of $\mathrm{PSL}(2, \mathbb{C})$ such that the limit set is a finite set. It is empty or it consists of 1 or 2 points.

Equivalently, a Kleinian group is elementary if it is virtually abelian; that is, it has an abelian subgroup of finite index.



(a) elliptic (b) hyperbolic (c) loxodromic (d) parabolic

Generalize to higher dimensions

Discrete subgroups of isometries of hyperbolic 3-space $\mathbb{H}^3_{\mathbb{R}}$ (discrete groups of conformal automorphisms of the sphere at infinity)	$\cong$	Discrete groups of holomorphic transformations of the complex projective line $\mathbb{CP}^1 \cong \mathbb{S}^2$ acting with nonempty region of discontinuity.
$\downarrow$		$\downarrow$
Conformal Kleinian Groups		Complex Kleinian Groups

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Preliminaries

The complex projective plane $\mathbb{CP}^2$ is defined as

$$\mathbb{CP}^2 = \left(\mathbb{C}^3 \setminus \{0\}\right)/\mathbb{C}^*,$$

where $\mathbb{C}^*$ acts by the usual scalar multiplication. Let

$$[\,]: \mathbb{C}^3 \setminus \{0\} \to \mathbb{CP}$$

be the quotient map. We denote the projectivization of the point $x = (x_1, x_2, x_3) \in \mathbb{C}^3$ by $[x] = [x_1 : x_2 : x_3]$. We denote by e_1, e_2, e_3 the projectivization of the canonical base of $\mathbb{C}^3$.

Preliminaries

Let $\mathrm{GL}(3,\mathbb{C}) \subset \mathcal{M}_3(\mathbb{C})$ be the subgroup of matrices with determinant not equal to 0. The group of biholomorphic automorphisms of $\mathbb{CP}^2$ is given by

$$\mathrm{PSL}(3,\mathbb{C}) := \mathrm{GL}(3,\mathbb{C}) / \{\text{scalar matrices}\}.$$

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We denote the upper triangular subgroup of $\mathrm{PSL}(3,\mathbb{C})$ by

$$U_+ = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \middle| \, a_{11}a_{22}a_{33} = 1, \ a_{ij} \in \mathbb{C} \right\}.$$

Preliminaries

As in the case of automorphisms of $\mathbb{CP}^1$, we classify the elements of $\mathrm{PSL}(3, \mathbb{C})$ in three classes: elliptic, parabolic and loxodromic. However, unlike the classical case, there are several subclasses in each case. We now give a quick summary of the subclasses of elements we will be using.

Classification of elements of $PSL(3, \mathbb{C})$

An element $g \in PSL(3, \mathbb{C})$ is said to be:

- Elliptic if it has a diagonalizable lift in $SL(3, \mathbb{C})$ such that every eigenvalue has norm 1.
- Parabolic if it has a non-diagonalizable lift in $SL(3, \mathbb{C})$ such that every eigenvalue has norm 1.

Classification of elements of $\mathrm{PSL}(3,\mathbb{C})$

- Loxodromic if it has a lift in $\mathrm{SL}(3,\mathbb{C})$ with an eigenvalue of norm distinct of 1. Furthermore, we say that g is:
  - Loxo-parabolic

$$\mathbf{h} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}, \ |\lambda| \neq 1.$$

  - A complex homothety, $\mathbf{h} = \mathrm{Diag}\left(\lambda, \lambda, \lambda^{-2}\right)$, $|\lambda| \neq 1$.
  - A rational (resp. irrational) screw, $\mathbf{h} = \mathrm{Diag}\left(\lambda_1, \lambda_2, \lambda_3\right)$, $|\lambda_1| = |\lambda_2| \neq |\lambda_3|$ and $\lambda_1\lambda_2^{-1} = e^{2\pi i\theta}$ with $\theta \in \mathbb{Q}$ (resp. $\theta \in \mathbb{R}\setminus\mathbb{Q}$).
  - Strongly loxodromic, $\mathbf{h} = \mathrm{Diag}\left(\lambda_1, \lambda_2, \lambda_3\right)$, where $\{|\lambda_1|, |\lambda_2|, |\lambda_3|\}$ are pairwise different.

The Kulkarni Limit Set

From now on, let $\Gamma \subset \mathrm{PSL}(3, \mathbb{C})$ be a discrete subgroup acting on $\mathbb{CP}^2$.

Definition

- *Let $L_0(\Gamma)$ be the closure of the set of points in $\mathbb{CP}^n$ with infinite isotropy group.*
- *Let $L_1(\Gamma)$ be the closure of the set of cluster points of orbits of points in $\mathbb{CP}^n \setminus L_0(\Gamma)$.*
- *Let $L_2(\Gamma)$ be the closure of the set of cluster points of compact sets of $\mathbb{CP}^n \setminus (L_0(\Gamma) \cup L_1(\Gamma))$.*

$$\Lambda_{Kul}(\Gamma) = \overline{L_0(\Gamma) \cup L_1(\Gamma) \cup L_2(\Gamma)}, \quad \Omega_{Kul}(\Gamma) = \mathbb{CP}^n \setminus \Lambda_{Kul}(\Gamma).$$

Limit sets for complex Kleinian groups

The Kulkarni limit set $\Lambda_{\mathrm{Kul}}(\Gamma)$ in $\mathbb{CP}^2$ is made up of points and complex projective lines. It contains 1, 2, 3, 4 or ∞ lines in general position.

The equicontinuity region for Γ, denoted $\mathrm{Eq}(\Gamma)$, is defined to be the set of points $z \in \mathbb{CP}^n$ for which there is an open neighborhood U of z such that Γ restricted to U is a normal family.
Γ acts properly and discontinuously on $\mathrm{Eq}(\Gamma)$, and

$$\mathrm{Eq}(\Gamma) \subset \Omega_{\mathrm{Kul}}(\Gamma).$$

Elementary Complex Kleinian Group

How can we define *elementary* complex Kleinian groups?

- Discrete subgroups of $\mathrm{PSL}(3, \mathbb{C})$ such that its Kulkarni limit set contains a finite number of lines.

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- Discrete subgroups of $\mathrm{PSL}(3,\mathbb{C})$ with reducible action.

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- Discrete subgroups of $\mathrm{PSL}(3,\mathbb{C})$ with reducible action.
- Discrete solvable subgroups of $\mathrm{PSL}(3,\mathbb{C})$.

Solvable groups

- If $g, h \in G$, we define the commutator as $[g, h] = g^{-1}h^{-1}gh$.
- We define the commutator subgroup as

$$[G, G] = \{[g, h] \mid g, h \in G\} .$$

- The derived series of G is given by

$$G^{(0)} = G, \quad G^{(i+1)} = \left[G^{(i)}, G^{(i)}\right] .$$

- We say that G is solvable if, for some $n \geq 0$, we have $G^{(n)} = \{id\}$.

Solvable groups: Examples

- The infinite dihedral group is solvable

$$\mathrm{Dih}_\infty = \langle \mathrm{Rot}_\infty,\ z \mapsto -z \rangle.$$

- Any triangular group is solvable, with solvability length at most 3.
- The special orthogonal group is not solvable,

$$\left\{ \begin{bmatrix} a & -c \\ c & \bar{a} \end{bmatrix} \middle| |a|^2 + |b|^2 = 1 \right\} \subset \mathrm{PSL}(2, \mathbb{C})$$

- Cyclic groups, abelian groups.

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Background

Action on $\mathbb{CP}^2$

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Action on $\mathbb{CP}^2$

Irreducible

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Irreducible ✓

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Reducible action $\Big\{$

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$$\text{Reducible action} \begin{cases} \text{Solvable} \\ \end{cases}$$

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Main Result - Part 1: The Dynamics

Theorem (2020)

Let $\Gamma \subset PSL(3, \mathbb{C})$ be a triangularizable complex Kleinian group such that its Kulkarni limit set does not consist of exactly four lines in general position. Then there exists a non-empty open region $\Omega_\Gamma \subset \mathbb{CP}^2$ such that

1. *Ω_Γ is the maximal open set where the action is proper and discontinuous.*
2. *Ω_Γ is homeomorphic to one of the following regions: $\mathbb{C}^2$, $\mathbb{C}^2 \setminus \{0\}$, $\mathbb{C} \times (\mathbb{H}^+ \cup \mathbb{H}^-)$ or $\mathbb{C} \times \mathbb{C}^*$.*
3. *Γ is finitely generated and rank$(\Gamma) \leq 4$.*

Main Result - Part 1: The Dynamics

Theorem (2020)

Let $\Gamma \subset PSL(3, \mathbb{C})$ be a triangularizable complex Kleinian group such that its Kulkarni limit set does not consist of exactly four lines in general position. Then

(iv) *The group Γ can be written as*

$$\Gamma = \Gamma_p \rtimes \underbrace{\langle \eta_1 \rangle \rtimes \ldots \rtimes \langle \eta_m \rangle}_{\text{loxo-parabolic}} \rtimes \underbrace{\langle \gamma_1 \rangle \rtimes \ldots \rtimes \langle \gamma_n \rangle}_{\text{strongly loxodromic}}$$

where Γ_p is the subgroup of Γ consisting of all the parabolic elements of Γ.

(v) *The group Γ leaves a full flag invariant.*

Main Result - Part 1: The Dynamics

Theorem (2020)

Let $\Gamma \subset PSL(3,\mathbb{C})$ be a solvable complex Kleinian group such that its Kulkarni limit set does not consist of exactly four lines in general position. Then

(iv) *The group Γ can be written as*

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Notation

1. Torus groups,

$$\mathcal{T}(W) = \left\{ \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \middle| (x,y) \in W \right\}$$

   where $W \subset \mathbb{C}^2$ is a additive subgroup.

2. Dual torus groups,

$$\mathcal{T}^*(W) = \left\{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle| (x,y) \in W \right\}$$

   where $W \subset \mathbb{C}^2$ is a discrete additive subgroup with $r(W) \leq 2$.

Up to conjugation and a finite index subgroup, there are 16 types of non-cyclic, discrete solvable subgroups containing loxodromic elements.

We study these groups by exploring the parabolic part of the group. If $\Gamma \subset \mathrm{PSL}(3, \mathbb{C})$ is a group, we denote by $\Gamma_p \subset \Gamma$ the subgroup generated by all parabolic elements of Γ. We will say that Γ_p is the parabolic part of Γ. Parabolic Part.

Barrera, W., Cano, A., Navarrete, J. P., & Seade, J. (2022). Discrete parabolic groups in PSL (3, C). Linear Algebra and its Applications, 653, 430-500.

Main Result - Part 2: The Representations

Theorem (2023)

Main Theorems...

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Ideas behind the proof

The parabolic part is described in

Barrera, W., Cano, A., Navarrete, J. P., & Seade, J. (2022). Discrete parabolic groups in PSL (3, C). Linear Algebra and its Applications, 653, 430-500.

(v) Invariant Flag

Borel fixed point theorem: Let G be a connected solvable group acting morphically on a non-empty complete variety V. Then G has a fixed point in V. Morphical action

Applying this theorem to the Zariski closure of Γ yields that Γ is virtually triangularizable. Namely, Γ has a finite index subgroup such that, up to conjugation, is upper triangular.

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This proves (v).

Conclusions (i)-(iv) are proved together.

Restrictions on the Elements of a Non-commutative Group

Proposition

Let $\Gamma \subset U_+$ be a discrete subgroup. Let $\gamma \in \Gamma$ be an irrational screw $\gamma = Diag(\beta^{-2}e^{-6\pi i\theta}, \beta e^{4\pi i\theta}, \beta e^{2\pi i\theta})$, for some $|\beta| \neq 1$ and $\theta \in \mathbb{R} \setminus \mathbb{Q}$, then Γ is commutative.

Proposition

Let $\Gamma \subset U_+$ be a non-commutative, torsion-free discrete subgroup, then Γ cannot contain a type I complex homothety.

The Core of a Group

The core of Γ is an important purely parabolic subgroup of a complex Kleinian group Γ which determines the dynamics of Γ.

Proposition

The elements of $Core(\Gamma)$ *have the form*

$$g_{x,y} = \begin{bmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

for some $x, y \in \mathbb{C}$.

The Core of a Group

It is straightforward to verify that

$$\Lambda_{\text{Kul}}(\text{Core}(\Gamma)) = \bigcup_{g_{x,y} \in \text{Core}(\Gamma)} \overleftrightarrow{e_1, [0 : -y : x]}$$

We denote this pencil of lines by $\mathcal{C}(\Gamma) = \Lambda_{\text{Kul}}(\text{Core}(\Gamma))$.

Proposition

Let $\Gamma \subset U_+$ be a discrete group, then every element of Γ leaves $\mathcal{C}(\Gamma)$ invariant.

Commutativity

$$\begin{cases} \Gamma \text{ is not commutative} \\ \\ \Gamma \text{ is commutative} \end{cases}$$

Theorem

Let $\Gamma \subset U_+$ be a non-commutative, torsion free, complex Kleinian group, then

$$\Gamma = Core(\Gamma) \rtimes \langle \xi_1 \rangle \rtimes \ldots \rtimes \langle \xi_r \rangle \rtimes \rtimes \langle \eta_1 \rangle \rtimes \ldots \rtimes \langle \eta_m \rangle \rtimes \langle \gamma_1 \rangle \rtimes \ldots \rtimes \langle \gamma_n \rangle.$$

Furthermore, if $k = rank\,(Core(\Gamma))$ then $k + r + m + n \leq 4$.

Parabolic | Loxodromic

$$\begin{bmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \alpha & x & y \\ 0 & \beta & z \\ 0 & 0 & \beta \end{bmatrix} \qquad \begin{bmatrix} \alpha & x & y \\ 0 & \beta & z \\ 0 & 0 & \gamma \end{bmatrix}$$

$z \neq 0$ | $\alpha \neq \beta$, $z \neq 0$ | $\beta \neq \gamma$

Loxo-parabolic | Strongly loxodr

$\mathrm{Core}(\Gamma)$ | $A \setminus \mathrm{Ker}(\Gamma)$ | $\mathrm{Ker}(\lambda_{23}) \setminus A$

Morphisms λ

Let $\lambda_{12}, \lambda_{23}, \lambda_{13} : (U_+, \cdot) \to (\mathbb{C}^*, \cdot)$ be the group morphisms given by

$$\lambda_{12}\left([\alpha_{ij}]\right) = \alpha_{11}\alpha_{22}^{-1}$$
$$\lambda_{23}\left([\alpha_{ij}]\right) = \alpha_{22}\alpha_{33}^{-1}$$
$$\lambda_{13}\left([\alpha_{ij}]\right) = \alpha_{11}\alpha_{33}^{-1}.$$

Strategy of the proof:

- Decomposition of Γ in terms of $\mathrm{Ker}(\lambda_{23})$.
- Decompose $\mathrm{Ker}(\lambda_{23})$ in terms of $\mathrm{Ker}(\lambda_{12})$.
- Decompose $A = \mathrm{Ker}(\lambda_{12}) \cap \mathrm{Ker}(\lambda_{23})$ in terms of $\mathrm{Ker}(\Gamma)$

Rank

Theorem (Bestvina, Kapovich, Kleiner)

Let Γ be a group acting properly and discontinuously on a contractible manifold of dimension m, then $obdim(\Gamma) \leq m$.

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In our case, it can be re-stated as:

Theorem

Let $\Gamma \subset U_+$ be a non-commutative, torsion free, complex Kleinian group acting properly and discontinuously on a simply connected domain $\Omega \subset \mathbb{CP}^2$, then $k + r + m + n \leq 4$.

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Find a simply connected domain $\Omega \subset \mathbb{CP}^2$ where Γ acts properly and discontinuously, and then apply the theorem. In some cases, we write the explicit decomposition of Γ and verify that rank$(\Gamma) \leq 4$.

Some cases

Denote $\Sigma = \Pi(\Gamma)$. If Σ is discrete and $\mathrm{Ker}(\Gamma)$ is finite. If $|\Lambda(\Sigma)| \neq 2$, let

$$\Omega = \left(\bigcup_{z \in \Omega(\Sigma)} \overleftrightarrow{e_1, z} \right) \setminus \{e_1\}.$$

We know that Γ acts properly and discontinuously on Ω. If $|\Lambda(\Sigma)| = 0, 1$ or ∞, then each connected component of Ω is simply connected, since they are respectively homeomorphic to $\mathbb{CP}^2$, $\mathbb{C}^2$ or $\mathbb{C} \times \mathbb{H}$. By the theorem, it follows $k + r + m + n \leq 4$

For non-commutative Γ, using these ideas, we have constructed an open subset $\Omega_\Gamma \subset \mathbb{CP}^2$ such that the orbits of every compact set $K \subset \Omega_\Gamma$ accumulate on $\mathbb{CP}^2 \setminus \Omega_\Gamma$. Thus we can define a limit set for the action of Γ by $\Lambda_\Gamma := \mathbb{CP}^2 \setminus \Omega_\Gamma$. This limit set describes the dynamics of Γ, and the open region Ω_Γ satisfies (i) and (ii).

Also, we prove that $\text{rank}(\Gamma) \leq 4$. This verifies (iii).

Commutative groups

Theorem (Barrera, Cano, Navarrete, Seade)

Let $\Gamma \subset U_+$ be a commutative group, then Γ is conjugate in $PSL(3, \mathbb{C})$ to a subgroup of one of the following Abelian Lie Groups:

- $$C_1 = \left\{ \left(\begin{array}{ccc} \alpha^{-2} & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{array} \right) \middle| \alpha \in \mathbb{C}^*, \beta \in \mathbb{C} \right\}.$$
- $$C_2 = \left\{ Diag\left(\alpha, \beta, \alpha^{-1}\beta^{-1}\right) \middle| \alpha, \beta \in \mathbb{C}^* \right\}.$$
- $$C_3 = \left\{ \left(\begin{array}{ccc} 1 & 0 & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{array} \right) \middle| \beta, \gamma \in \mathbb{C} \right\}.$$

Commutative groups

Theorem (Barrera, Cano, Navarrete, Seade)

- $$C_4 = \left\{ \left(\begin{array}{ccc} 1 & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \middle| \beta, \gamma \in \mathbb{C} \right\}.$$

- $$C_5 = \left\{ \left(\begin{array}{ccc} 1 & \beta & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{array} \right) \middle| \beta, \gamma \in \mathbb{C} \right\}.$$

Case 1: Form

Proposition

Let $\Gamma \subset U_+$ be a commutative subgroup such that each element of Γ has the form C_1. Then there exists an additive subgroup $W \subset (\mathbb{C}, +)$, and a group morphism $\mu : (W, +) \to (\mathbb{C}^*, \cdot)$ such that

$$\Gamma = \Gamma_{W,\mu} = \left\{ \begin{bmatrix} \mu(w)^{-2} & 0 & 0 \\ 0 & \mu(w) & w\mu(w) \\ 0 & 0 & \mu(w) \end{bmatrix} \middle| w \in W \right\}.$$

Case 1: Discreteness and Rank

Proposition

Let $\Gamma = \Gamma_{W,\mu} \subset U_+$ be a group as described in previous proposition. Γ is discrete if and only if rank$(W) \leq 3$ and the morphism μ satisfies the following condition:

- *Whenever we have a sequence $\{w_k\} \in W$ of distinct elements such that $w_k \to 0$, either $\mu(w_k) \to 0$ or $\mu(w_k) \to \infty$.*

Case	Conditions
C1.1	$\mu(W)$ has rational rotations and W is discrete.
C1.2	$\mu(W)$ has rational rotations and W is not discrete.
C1.3	$\mu(W)$ has no rational rotations but has irrational rotations, and W is discrete.
C1.4	$\mu(W)$ has no rational or irrational rotations, and W is discrete.
C1.5	$\mu(W)$ has no rational rotations but has irrational rotations, and W is not discrete.
C1.6	$\mu(W)$ has no rational or irrational rotations, and W is not discrete.

Case 1: Kulkarni Limit Set

Theorem

Let $\Gamma \subset PSL(3, \mathbb{C})$ be a commutative discrete group having the form given previous proposition, then

$$\Lambda_{Kul}(\Gamma) = \begin{cases} \overleftrightarrow{e_1, e_2}, & \begin{cases} \text{Cases C1.3 or C1.4, with condition} \\ \text{Case C1.1} \end{cases} \\ \{e_1\} \cup \overleftrightarrow{e_2, e_3}, & \text{Cases C1.5 or C1.6 no condition } \mathbf{(F)}. \\ \overleftrightarrow{e_1, e_2} \cup \overleftrightarrow{e_2, e_3}, & \begin{cases} \text{Cases C1.5 or C1.6, with condition} \\ \text{Case C1.2} \end{cases} \end{cases}$$

Case 2: Form

Proposition

Let $\Gamma \subset U_+$ *be a commutative subgroup such that each element of* Γ *has the form* $Diag(\alpha, \beta, \alpha^{-1}\beta^{-1})$. *Then there exist two multiplicative subgroups* $W_1, W_2 \subset (\mathbb{C}^*, \cdot)$ *such that*

$$\Gamma = \Gamma_{W_1,W_2} = \{Diag(w_1, w_2, 1) \,|\, w_1 \in W_1,\ w_2 \in W_2\}\,.$$

Case 2: Rank

Proposition

Let $\Gamma \subset U_+$ be a diagonal discrete group such that every element has the form $\gamma = Diag(w_1, w_2, 1)$. Then $rank(\Gamma) \leq 2$.

Case 2

If $\alpha^n = \beta^m$ for some $n, m \in \mathbb{Z}$:

[D1] $L_0(\Gamma) \cup L_1(\Gamma) = \overleftrightarrow{e_1, e_2} \cup \{e_3\}$, if $|\alpha| > 1 > |\beta|$ or $|\alpha| < 1 < |\beta|$.

[D2] $L_0(\Gamma) \cup L_1(\Gamma) = \overleftrightarrow{e_1, e_2} \cup \{e_3\}$, if $|\alpha| > |\beta| > 1$ or $|\alpha| < |\beta| < 1$.

If there are no integers n, m such that $\alpha^n = \beta^m$:

[D3] $L_0(\Gamma) \cup L_1(\Gamma) = \{e_1, e_2, e_3\}$, if $|\alpha| > 1 > |\beta|$ or $|\alpha| < 1 < |\beta|$.

[D4] $L_0(\Gamma) \cup L_1(\Gamma) = \{e_1, e_2, e_3\}$, if $|\alpha| > |\beta| > 1$ or $|\alpha| < |\beta| < 1$.

[D5] $L_0(\Gamma) \cup L_1(\Gamma) = \overleftrightarrow{e_1, e_2} \cup \overleftrightarrow{e_2, e_3}$, if β is an irrational rotation.

Case 2: Kulkarni Limit Set

Theorem

Let $\Gamma_{\alpha,\beta} \subset U_+$ be a discrete group containing loxodromic elements, then

1. $\Lambda_{Kul}(\Gamma) = \overleftrightarrow{e_1, e_2} \cup \{e_3\}$ *in Cases* **[D1]** *and* **[D2]**.
2. $\Lambda_{Kul}(\Gamma) = \{e_1, e_2, e_3\}$ *in Cases* **[D3]** *and* **[D4]**.
3. $\Lambda_{Kul}(\Gamma) = \overleftrightarrow{e_1, e_2} \cup \overleftrightarrow{e_2, e_3}$ *in Case* **[D5]**.

Commutative case: Proof of the Main Theorem

If Γ is commutative, it is conjugate to a sugroup of the Lie groups C_1 or C_2. In this setting, the region $\Omega_{\mathrm{Kul}}(\Gamma)$ satisfies conclusions (i) and (ii) as a consequence of the previous heorems. Again, $\mathrm{rank}(\Gamma) \leq 4$, this proves conclusion (iii).

On the other hand, $\Gamma \cong \mathbb{Z}^r$ with $r = \mathrm{rank}(\Gamma)$, and then we can write Γ as a trivial semidirect product of copies of $\mathbb{Z}$, thus verifying conclusion (iv).

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A First Generalization ✓

Theorem

Let $\Gamma \subset PSL(3,\mathbb{C})$ be a solvable complex Kleinian group such that its Kulkarni limit set does not consist of exactly four lines in general position. Let $\Gamma_0 \subset \Gamma$ be a virtually triangularizable finite index subgroup. If Γ_0 is commutative then there exists a non-empty open region $\Omega_\Gamma \subset \mathbb{CP}^2$ such that

1. *Ω_Γ is the maximal open set where the action is proper and discontinuous.*
2. *Ω_Γ is homeomorphic to one of the following regions: $\mathbb{C}^2$, $\mathbb{C}^2 \setminus \{0\}$, $\mathbb{C} \times (\mathbb{H}^+ \cup \mathbb{H}^-)$ or $\mathbb{C} \times \mathbb{C}^*$.*
3. *Up to a finite index subgroup, the group Γ leaves a full flag invariant.*

Thank you

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A Group Acting Morphically

Definition

Let G be an algebraic group, V a variety, and let $\alpha : G \times V \to V$ be an action of the group G in V, $(g, x) \mapsto gx = \alpha(g, x)$. One says that G acts morphically on V if the action α satisfies the following axioms:

1. $\alpha(e, x) = x$, *for any* $x \in V$, *where* $e \in G$ *is the identity element.*
2. $\alpha(g, hx) = \alpha(gh, x)$ *for any* $g, h \in G$ *and* $x \in V$.

Solvable groups are virtually triangular