Lattice Diversities

Mauricio Toledo-Acosta

Departamento de Matemáticas
Universidad de Sonora

Bryant, D., Felipe, R., Toledo-Acosta, M., & Tupper, P. (2020). Lattice Diversities. arXiv preprint arXiv:2010.11442.

https://arxiv.org/abs/2010.11442

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The Metric Tight-span

- Hyperconvex metric spaces were defined by Aronszajn and Panitchpakdi in 1956 [1] as part of a programme to generalise the Hahn–Banach theorem to more general metric spaces.

Hyperconvex spaces

- Isbell and Dress showed that, for every metric space, there exists an essentially unique *minimal* hyperconvex space into which that space could be embedded, called the tight-span or injective envelope.

[1] Extension of uniformly continuous transformations and hyperconvex metric spaces, Pacific J. Math. 6 (1956)

The Metric Tight-span

Dress et al. [16] coined the term T-theory for the field of discrete mathematics devoted to the combinatorics of the tight span and related constructions. Contributions to T-theory include

- Optimal graph realisations of metrics [11,13,30].
- links with tropical geometry and hyperdeterminants [9,31]
- classification of finite metrics [11,40]

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This section is based on

Bryant, D., Tupper, P. F. (2012). Hyperconvexity and tight-span theory for diversities. Advances in Mathematics, 231(6), 3172-3198.

Diversities

Diversities are an extension of metric spaces where instead of the non-negative function being defined on pairs of points, it is defined on arbitrary finite sets of points.

Diversity (v1)

Let $\mathcal{P}_{\text{fin}}(X)$ be the set of all finite subsets of a set X. A diversity is a set X with a function $\delta\colon \mathcal{P}_{\text{fin}}(X) \to \mathbb{R}_{\geq 0}$ satisfying

1. $\delta(A) = 0$ if and only if $|A| \leq 1$.
2. $\forall\, A, B, C \in \mathcal{P}_{\text{fin}}(X)$ such that $B \neq \varnothing$

$$\delta(A \cup C) \leq \delta(A \cup B) + \delta(B \cup C).$$

Diversities

Diversities are an extension of metric spaces where instead of the non-negative function being defined on pairs of points, it is defined on arbitrary finite sets of points.

Diversity (v2)

Let $\mathcal{P}_{\text{fin}}(X)$ be the set of all finite subsets of a set X. A diversity is a set X with a function $\delta\colon \mathcal{P}_{\text{fin}}(X) \to \mathbb{R}_{\geq 0}$ satisfying

1. $\delta(A) = 0$ if and only if $|A| \leq 1$.
2. $\forall\ A, B \in \mathcal{P}_{\text{fin}}(X)$ such that $A \subseteq B$, $\delta(A) \leq \delta(B)$
3. If $A \cap B \neq \varnothing$ then $\delta(A \cup B) \leq \delta(A) + \delta(B)$.

Examples of diversities

1. **Diameter diversity:** Let (X, d) be a metric space. For all finite $A \subset X$ let
$$\delta_{\text{diam}}(A) = \max_{a,b \in A} d(a, b).$$
2. ℓ_1 **diversity:** For any finite $A \subset \mathbb{R}^m$ let
$$\delta(A) = \sum_{i=1}^{m} \max_{a,b} \{|a_i - b_i| \mid a, b \in A\}.$$

Examples of diversities

1. **Steiner diversity:** Let (X, d) be a metric space. For each finite $A \subset X$ denote by $\delta(A)$ the minimum length of a Steiner tree connecting elements in A. Steiner problem

2. **Hypergraph Steiner diversity:** Let $H = (X, E)$ be a hypergraph and let $w : E \to \mathbb{R}_{\geq 0}$ be a non-negative weight function. Given $A \subset X$ let

$$\delta(A) = \min_{e \in E'} w(e),$$

the minimum is taken over all subsets $E' \subset E$ such that the sub-hypergraph induced by E' is connected and includes A.

Metrics and Diversities

Proposition

Let (X, δ_1), (X, δ_2) be two diversities, then

- $\delta = \delta_1 + \delta_2$ *is a diversity.*
- $\delta = r\delta_1$, *for* $r > 0$, *is a diversity.*
- $\max\{\delta_1, \delta_2\}$ *is a diversity.*
- $\delta = \dfrac{\delta}{1+\delta}$ *is a diversity.*

Proposition

Let (X, δ) be a diversity, then

$$d(x, y) = \delta(\{x, y\})$$

is a metric on X.

Tight Span of a Diversity

Tight Span of a Diversity

Let (X, δ) be a diversity. Let

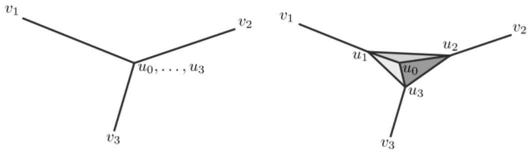
$$P_X = \left\{ f : \mathcal{P}_{\text{fin}}(X) \to \mathbb{R} \;\middle|\; \begin{array}{l} f(\emptyset) = 0, \\ \sum_{A \in \mathcal{A}} f(A) \geq \delta\left(\bigcup_{A \in \mathcal{A}} A\right) \\ \forall \text{ finite } \mathcal{A} \subset \mathcal{P}_{\text{fin}}(X) \end{array} \right\}$$

We write $f \leq g$ if $f(A) \leq g(A)$ for all $A \in \mathcal{P}_{\text{fin}}(X)$. The tight span of (X, δ) is the set of minimal elements in P_X under the partial order $\leq$.

If $|X| = n$ then T_X can be regarded as a subset of $\mathbb{R}^{2^n - 1}$.

An example

If $X = \{1, 2, 3\}$, the tight-span is



depending on whether $2d_{123} \leq d_{12} + d_{23} + d_{13}$ or $2d_{123} > d_{12} + d_{23} + d_{13}$.

The tight-span is a diversity

Theorem

Let (X, δ) be a diversity, let $\delta_T : \mathcal{P}_{\text{fin}}(T_X) \to \mathbb{R}$ be the function defined by $\delta_T(\emptyset) = 0$ and

$$\delta_T(F) = \sup_{\mathcal{A} \subset \mathcal{P}_{\text{fin}}(X)} \left\{ \delta\left(\bigcup_{A \in \mathcal{A}} A \right) - \sum_{A \in \mathcal{A}} \inf_{f \in F} f(A) \right\}$$

for all $F \in \mathcal{P}_{\text{fin}}(T_X)$. Then (T_X, δ_T) is a diversity.

Diversity embedding

Diversity embedding

Let (Y_1, δ_1) and (Y_2, δ_2) be two diversities. A map $\pi : Y_1 \to Y_2$ is an embedding if

1. π is injective.
2. $\forall A \subset \mathcal{P}_{\text{fin}}(Y_1)$ we have $\delta_1(A) = \delta_2(\pi(A))$.

An isomorphism is a surjective embedding between two diversities.

Embedding a diversity into its tight-span

- Let (X, δ) be a diversity. For each $x \in X$ define the function $h_x : \mathcal{P}_{\text{fin}}(X) \to \mathbb{R}_{\geq 0}$ by

$$h_x(A) = \delta\left(A \cup \{x\}\right)$$

for all $A \in \mathcal{P}_{\text{fin}}(X)$.

- For all $x \in X$, $h_x \in T_X$.
- Let $\kappa : X \to T_X$ given by $\kappa(x) = h_x$.

Theorem

The map κ is an embedding from (X, δ) into (T_X, δ_T).

Injectivity

Injective Diversity

A diversity (X, δ) is injective if it satisfies: given any pair of diversities (Y_1, δ_1), (Y_2, δ_2), an embedding $\pi : Y_1 \to Y_2$, and a non-expansive map $\phi : Y_1 \to X$ there is a non-expansive map $\psi : Y_2 \to X$ such that $\phi = \psi \circ \pi$.

Hyperconvexity

Hyperconvex Diversity

A diversity (X, δ) is hyperconvex if $\forall\ r : \mathcal{P}_{\text{fin}}(X) \to X$ such that

$$\delta\left(\bigcup_{A \in \mathcal{A}} A\right) \leq \sum_{A \in \mathcal{A}} r(A).$$

for all $\mathcal{A} \in \mathcal{P}_{\text{fin}}(X)$ there is $z \in X$ such that $\delta(\{z\} \cup Y) \leq r(Y)$ for all $Y \in \mathcal{P}_{\text{fin}}(X)$.

Main results of the classic case

Theorem

A diversity (X, δ) is injective if and only if it is hyperconvex.

Theorem

For any diversity (X, δ), the tight-span (T_X, δ_T) is hyperconvex.

Connection with the metric case

$$
\begin{array}{ccc}
(X, d) & \xrightarrow{\text{tight-span}} & (T_X^d, d_T) \\
\delta = \text{diam} \big\downarrow & & \big\downarrow \delta = \text{diam} \\
(X, \delta) & \xrightarrow{\text{tight-span}} & (T_X, \delta_T)
\end{array}
$$

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POSET

POSET

Let $P \neq \emptyset$ be a set and let $\leq$ be a relation satisfying for all $a, b, c \in P$

1. $a \leq a$
2. $a \leq b$ and $b \leq c$ implies $a \leq c$,
3. $a \leq b$ and $b \leq a$ implies $a = b$.

We say that $(P, \leq)$ is a POSET.

If there is an element $x \in P$ such that $x \leq y$ for all $y \in P$, we let $x = 0$ and say that P has a 0.

Examples of POSETS

- $(\mathbb{R}, \leq)$
- The power set of any non-empty set X, $(2^X, \subset)$
- The set $\mathbb{N}$ equipped with the relation of divisibility, $(\mathbb{N}, |)$.
- For any set X and any POSET P, the set of functions $f : X \to P$ with the usual order.

Lattice: as a POSET

Lattice

A lattice is a non-empty poset $(L, \leq)$ in which every two elements have a unique supremum (also called a least upper bound) and a unique infimum (also called a greatest lower bound).

Lattice: as a POSET

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There are two binary operations defined on L called meet (designated by $\wedge$) and join (designated by $\vee$). They are given by the infinum and supremum respectively.

Lattice: as a POSET

Lattice

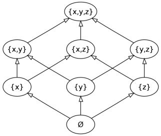
A lattice is a non-empty poset $(L, \leq)$ in which every two elements have a unique supremum (also called a least upper bound) and a unique infimum (also called a greatest lower bound).

There are two binary operations defined on L called meet (designated by $\wedge$) and join (designated by $\vee$). They are given by the infinum and supremum respectively.

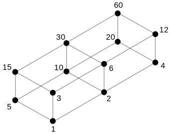
Lattices, being posets, may have a 0 element, which satisfies $0 \wedge x = 0$ and $0 \vee x = x$ for all $x \in L$.

Examples of Lattices

- $(\mathbb{N}, \leq)$. The supremum is given by the maximum, and the infimum by the mininum.
- For any set X, $(\mathcal{P}_{\text{fin}}(X), \subseteq)$. The supremum is given by the union, and the infimum by the intersection.



- $(\mathbb{N}, |)$. The supremum is given by the lcm, and the infimum by the GCD.



Lattice: As algebraic structure

Lattice

A lattice is an algebraic structure $(L, \vee, \wedge)$ consisting of a set L and two binary, commutative and associative operations $\vee$ and $\wedge$, called join and meet, satisfying the absorption laws:

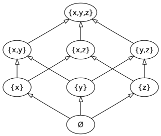
- $a \vee (a \wedge b) = a$
- $a \wedge (a \vee b) = a$

Lattices, may have a 0 element, which is the identitity of $\vee$, that is $0 \vee x = x$ for all $x \in L$.

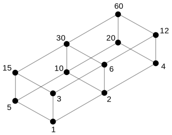
For $B \subseteq L$ we let $\bigvee B$ denote $\bigvee_{b \in B} b$; likewise for $\bigwedge B$.

Examples of Lattices

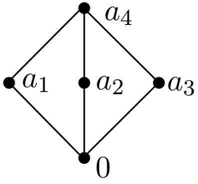
- $(\mathbb{N}, \min, \max)$.
- For any set X, the collection of all finite subsets of X, $(\mathcal{P}_{\text{fin}}(X), \cup, \cap)$.

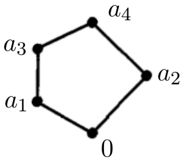


- $(\mathbb{N}, \text{lcm}, \text{GCD})$.



More examples of Lattices: Hasse Diagram





$J(L)$ and $A(L)$

Let L be a lattice.

- An element $a \in L$ is said to be join irreducible if $a \neq 0$ and $a = b \vee c$ implies $a = b$ or $a = c$.
- We denote by $J(L)$ the set of all join irreducible elements.
- For $a, b \in L$, we say that b covers a if $a < b$ and

$$\{c \mid a < c < b\} = \varnothing.$$

- If L has a least element 0, then the upper covers of 0 are called the atoms of L, and we denote them by $A(L)$. Note that

$$A(L) \subseteq J(L).$$

Lattice Diversity

Lattice Diversity (v2)

Let L be a lattice with 0 and $\delta : L \to \mathbb{R}_{\geq 0} pos$. (L, δ) is a lattice diversity if

1. $\delta(a) = 0$ if and only if $a \in A(L)$ or $a = 0$.
2. $a \leq b$ implies $\delta(a) \leq \delta(b)$ (monotonicity),
3. $a \wedge b \neq 0$ implies $\delta(a \vee b) \leq \delta(a) + \delta(b)$ (subadditivity on non-disjoint elements)

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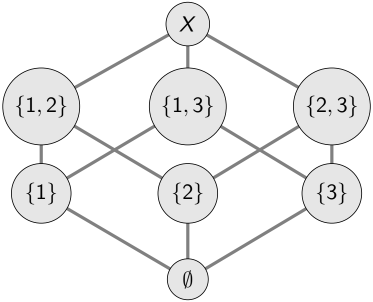
The triangle inequality does not imply monotonicity and subadditivity on non-disjoint elements. Example

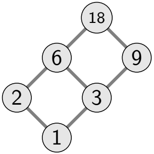
Examples

- **The classical case:** If $L = \mathcal{P}_{\text{fin}}(X)$, then we recover the original definition of classical diversities. In this case, the role of X is played by $A(L) = J(L)$.



Examples

- **Divisibility diversity:** $(\mathbb{N}, \text{lcm}, \text{GCD})$. The least element of L is 1 and $A(L)$ is the set of prime numbers.



Let $\Omega(n)$ be the Omega function which counts the total number of prime factors of n, counting multiplicity. Consider the function

$$\delta(n) = \begin{cases} \Omega(n), & n \text{ is not prime}, \\ 0, & n \text{ otherwise}. \end{cases}$$

(L, δ) is a diversity.

Basic properties of lattice diversities

Proposition

Let (L, δ) be a lattice diversity and let $x_1, \ldots, x_n \in L$ such that $x_i \wedge x_{i+1} \neq 0$ for all $i = 1, \ldots, n-1$. Then

$$\delta(x_1 \vee \ldots \vee x_n) \leq \delta(x_1) + \ldots + \delta(x_n).$$

Proposition

For a lattice diversity (L, δ), the diversity function δ satisfies the triangle inequality:

$$\delta(a \vee c) \leq \delta(a \vee b) + \delta(b \vee c),$$

for all $a, b, c \in L$ with $b \neq 0$.

Basic properties of lattice diversities

Proposition

For a lattice diversity (L, δ), define the function $d_\delta : A(L) \times A(L) \to \mathbb{R}_{\geq 0}$ by

$$d_\delta(a, b) = \delta(a \vee b)$$

then $(A(L), d_\delta)$ is a metric space.

Proposition

Let δ be a diversity on a lattice L, and define $d_{n,\delta} : X^n \to \Re^+$ by

$$d_{n,\delta}(a_1, ..., a_n) = \delta\left(a_1 \vee ... \vee a_n\right).$$

Then $d_{n,\delta}$ is an n-way distance on $A(L)$.

Diversities for particular classes of lattices

Proposition

Let L be any lattice with a 0 and let $\delta : L \to \mathbb{R}_{\geq 0}$ be

$$\delta(a) = \begin{cases} 0, & a \in A(L) \cup \{0\}, \\ 1, & \textit{otherwise.} \end{cases}$$

Then (L, δ) is a lattice diversity.

Diversities for particular classes of lattices

Proposition

Let L be a modular lattice sectionally of finite height and let $\delta_h : L \to \mathbb{R}_{\geq 0}$ be

$$\delta_h(a) = \begin{cases} 0, & a = 0, \\ h(a) - 1, & \textit{otherwise.} \end{cases}$$

Then (L, δ_h) is a strictly monotone lattice diversity.

Modular lattices
Lattices sectionally of finite height

Diversities for particular classes of lattices

Proposition

Let L be a lattice with a 0 and let $v : L \to \mathbb{R}_{\geq 0}$ be a positive sub-valuation with $v(0) = 0$. Let $\delta_v : L \to \mathbb{R}_{\geq 0}$ be given by

$$\delta_v(a) = \begin{cases} 0, & a \in A(L), \\ v(a), & \textit{otherwise}. \end{cases}$$

Then, (L, δ_v) is a strictly monotone lattice diversity.

Valuation on lattices

Finite Distributive Lattice Diversities

Every lattice diversity on a finite distributive lattice is isomorphic to a restriction of a classical diversity to a lattice of finite lower subsets of a set.

Proposition

Let (L, δ) be a lattice diversity where L is finite distributive lattice. For $A \subseteq J(L)$, we define

$$\hat{\delta}(A) = \begin{cases} \delta\left(\bigvee A\right), & \text{if } |A| \geq 2 \\ 0, & \text{otherwise.} \end{cases}$$

Then $(J(L), \hat{\delta})$ is a classical diversity whose restriction to $\mathcal{O}\left(J(L)\right)$ is isomorphic to (L, δ).

$\mathcal{O}(P)$ Distributive Lattices Birkhoff's representation theorem

The tight-span of a lattice diversity

The tight-span of a lattice diversity

Let (L, δ) be a lattice diversity. Let P_L denote the set of all functions $f : L \to \mathbb{R}_{\geq 0}$ satisfying

$$\sum_{b \in B} f(b) \geq \delta\left(\bigvee B\right),$$

for any finite subset $B \in \mathcal{P}_{\text{fin}}(L)$. Write $f \preceq g$ if $f(a) \leq g(a)\ \forall a \in L$. The tight-span of (L, δ) is the set $T_L \subseteq P_L$ of functions that are minimal under $\preceq$.

Properties of T_L

Proposition

Suppose that $f \in T_L$. Then

1. $f(a) \geq \delta(a)$ *for any* $a \in L$.
2. *Let* $a, b \in L$ *such that* $a \leq b$ *then* $f(a) \leq f(b)$. *That is,* f *is monotone.*
3. $f(a \vee c) \leq \delta(a \vee b) + f(b \vee c)$ *for any* $a, b, c \in L$, $b \neq 0$.
4. $f(a \vee b) \leq f(a) + f(b)$ *for any* $a, b \in L$. *That is,* f *is sub-additive.*

Embedding a lattice diversity into its tight span

For $x \in L$, let $h_x : L \to \mathbb{R}_{\geq 0}$ be the function given by

$$h_x(a) = \delta\left(x \vee a\right).$$

We define the map κ given by $\kappa(x) = h_x$.

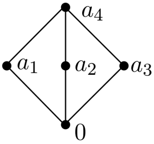
Theorem

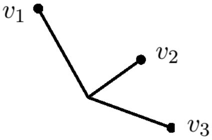
Let (L, δ) be a lattice diversity. The function $\kappa : (L, \leq) \to (P_L \cup \{\delta\}, \preceq)$ defined above is an order-preserving map. Also,

1. $\kappa(0) = \delta$.
2. $\kappa(a) \in P_L$ *if* $a \neq 0$.
3. $\kappa(a) \in T_L$ *if* $a \in A(L)$.
4. κ *restricted to* $A(L) \cup \{0\}$ *is injective.*
5. *If* δ *is strictly monotone, then* κ *is injective on* L.

Examples

- If (X, δ) is a classical diversity, then the lattice diversity $(\mathcal{P}_{\text{fin}}(X), \delta)$ has the same tight span as (X, δ) does in the classical theory of diversities.
- Let $L = \mathbf{M}_3$



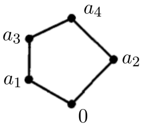


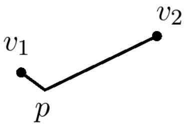
Any diversity is determined by $\alpha = \delta(a_4) > 0$.

Details

Examples

- Let $L = \mathbf{N}_5$





Any diversity on N_5 is determined by the two values

$$\alpha = \delta(a_3), \quad \beta = \delta(a_4),$$

We have

$$\begin{aligned} v_1 &= (0, 0, \beta, \alpha, \beta) = \kappa(a_1) \\ v_2 &= (0, \beta, 0, \beta, \beta) = \kappa(a_2) \\ p &= (0, \alpha, \beta - \alpha, \alpha, \beta) \end{aligned}$$

The obstacles for a T-theory

- Though many aspects of the theory of classical diversities carry over directly to lattice diversities, we were not able to develop much of the theory of tight spans in this new setting.
- In particular, we were not able to find a natural embedding of a lattice diversity into its tight span, or any object associated with it, such as $P_L \cup \{\delta\}$.
- The natural mapping κ from a lattice diversity to the set of real-valued functions on the lattice is not a lattice homomorphism.
- This motivates looking for other ways of defining the tight span of a lattice diversity.

Category Theory

- If we view a metric space as an object in the concrete category **Met** of metric spaces with non-expansive mappings, then the tight span of a metric space is its *injective hull*, which is the unique essential embedding of the metric space into an injective object.
- The analogous result is true for classical diversities in the category **Div**.
- In general, for any concrete category (which we specify by its objects and morphisms) if all objects have injective hulls then the injective hulls are a good candidate for an analogue of the tight span.

The Category of Lattice Diversities

- We define a category where our objects are lattice diversities and define morphisms to be maps $f\colon (L_1, \delta_1) \to (L_2, \delta_2)$ between lattices that satisfy
  - f is a lattice homomorphism,
  - $\delta_2(f(a)) \leq \delta_1(a)$, for all $a \in L$.
- Now a minimal requirement for objects in this category to have injective hulls is that each object be embeddable into some injective object.
- The injective objects of this category are the single point lattices with the trivial diversity on them.
- This implies that any lattice diversity with more than two points cannot have an injective hull.

Alternatives

- Seek a meet-free axiomatization of lattice diversities and take morphisms to be join-homomorphisms, since semilattices have a richer set of injective objects.
- Consider restricted classes of lattice diversities, such distributive lattice diversities. The injective objects in the category of distributive lattices are the complete Boolean lattices. However, Boolean lattices are isomorphic to the subalgebra of a power set by Stone's representation theorem. Thus, recapitulating that of the tight span theory for diversities

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Hyperconvex Spaces

Hyperconvex Space

A metric space X is said to be hyperconvex if it is convex and satisfies

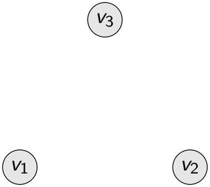
- Any two points x and y can be connected by the isometric image of a line segment of length equal to the distance between the points.
- If F is any familiy of closed balls with non-empty pairwise intersection, then there exists a point common to all the balls in F.

Examples: $\mathbb{R}$, $\mathbb{R}^2$ with the Manhattan distance, $\mathbb{R}$-trees.

Metric Tight-Span

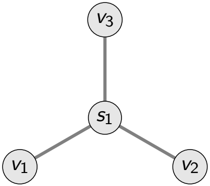
Steiner tree problem

Given N points in the plane, the goal is to connect them by lines of minimum total length in such a way that any two points may be interconnected by line segments either directly or via other points and line segments.



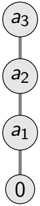
Steiner tree problem

Given N points in the plane, the goal is to connect them by lines of minimum total length in such a way that any two points may be interconnected by line segments either directly or via other points and line segments.



Examples of diversities

Non-equivalency of axioms



$$\begin{aligned}\delta(0) = \delta(a_1) &= 0\\ \delta(a_2) &= 2\\ \delta(a_3) &= 1\end{aligned}$$

δ satisfies condition (1) and the *triangle inequality* ($\delta(a \vee b) + \delta(b \vee c) \leq \delta(a \vee c)$ for all $a, b, c \in L$) but not monotonicity.

Lattice Diversity

Modular Lattices

Modular Lattice

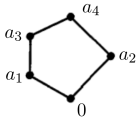
A lattice $(L, \vee, \wedge)$ which satisfies the modular law

$$a \wedge (b \vee c) = (a \wedge b) \vee c$$

for any $a, b, c \in L$ such that $c \leq a$ is called a modular lattice.

Examples:

- $(\mathcal{P}_{\text{fin}}(X), \cup, \cap)$ is modular.
- N_5 is not modular



$$a_1 \vee (a_2 \wedge a_3) = a_1 \vee 0 = a_1$$

$$(a_1 \vee a_2) \wedge a_3 = a_4 \vee a_3 = a_3$$

Finite Height Lattices

Height of a lattice

A lattice L is said to be of finite height iff there is a finite upper bound to the length of chains in L. The least such upper bound is called the height of L. A lattice L is said to be sectionally of finite height iff L has a least element 0, and for every $a \in L$, the interval $[0, a]$ is of finite height. In this case, the height of $[0, a]$ will be denoted by $h(a)$ and called the height of a.

Valuation on Lattices

A valuation on a lattice L is a function $v : L \to \mathbb{R}_{\geq 0}$ such that

$$v(a \wedge b) + v(a \vee b) = v(a) + v(b).$$

A sub-valuation is a function $v : L \to \mathbb{R}_{\geq 0}$ such that

$$v(a \wedge b) + v(a \vee b) \leq v(a) + v(b).$$

A valuation (resp. sub-valuation) is positive if $v(a) < v(b)$ whenever $a < b$.

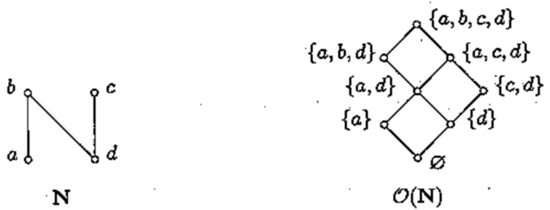
Examples:

- The height function is a positive valuation on modular lattices of sectionally finite height.
- The function $\log(x)$ on the lattice of divisibility of positive integers is a positive valuation.

Down-sets

Down-set

Let $(P, \leq)$ be a poset, we say that $Q \subseteq P$ is a down-set if, whenever $x \in Q$, $y \in P$ and $y \leq x$, we have $y \in Q$. The family of all down-sets of P is denoted $\mathcal{O}(P)$, it is a poset under the inclusion order.



Distributive Lattices

Distributive Lattices

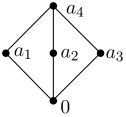
A lattice $(L, \vee, \wedge)$ which satisfies the distributive law

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

for any $a, b, c \in L$ is called a distributive lattice.

Examples:

- Every distributive lattice is modular.
- $(\mathcal{P}_{\mathrm{fin}}(X), \cup, \cap)$ is distributive.
- M_3 is not distributive



$$a_1 \wedge (a_2 \wedge a_3) = a_1 \vee a_4 = a_1$$

$$(a_1 \wedge a_2) \vee (a_1 \wedge a_3) = 0 \vee 0 = 0$$

Birkhoff's Representation Theorem

Theorem

Let L be a finite distributive lattice. Then the map

$$\eta(a) = \{x \in J(L) \mid x \leq a\} = J(L) \cap \downarrow a,$$

is an isomorphism of L onto the lattice of down-sets $\mathcal{O}(J(L))$. The inverse of the isomorphism is given by

$$\eta^{-1}(A) = \bigvee A,$$

for $A \subseteq J(L)$.

Details M_3

Any diversity on M_3 is determined by the value $\alpha = \delta(a_4) > 0$, since we have $\delta(0) = \delta(a_1) = \delta(a_2) = \delta(a_3) = 0$. Let $f : L \to \mathbb{R}_{\geq 0}$. f is determined by the values $f_i = f(a_i)$ and $f_0 = 0$. $f \in T_L$ iff

$$\begin{array}{lll} f_0, f_1, f_2, f_3 \geq 0, & f_4 \geq \alpha, & f_1 + f_2 \geq \alpha, \\ f_1 + f_3 \geq \alpha, & f_2 + f_3 \geq \alpha, & \end{array}$$

and for each f_i at least one of the inequalities it is in holds as an equality.

$$\begin{aligned} T_L = & \{(0, f_1, \alpha - f_1, \alpha - f_1, \alpha) \mid 0 \leq f_1 \leq \alpha/2\} \cup \\ & \{(0, \alpha - f_2, f_2, \alpha - f_2, \alpha) \mid 0 \leq f_2 \leq \alpha/2\} \cup \\ & \{(0, \alpha - f_3, \alpha - f_3, f_3, \alpha) \mid 0 \leq f_3 \leq \alpha/2\} \end{aligned}$$